Stability of general shock profiles – a novel weight function for the non-convex case

Christian Fries

Abstract. Consider a system

$$u_t + f(u)_x = \mu u_{xx} \tag{1}$$

of viscous conservation laws. Let ϕ be a profile for a shock wave associated with a simple eigenvalue λ of f', with end states $\phi(\pm\infty) = u_\pm$.

Then there are numbers ϵ_0, $\beta_0(u_-, u_+) > 0$ such that whenever $|u_+ - u_-| < \epsilon_0$ and $\bar{u}_0 \in L^1$ satisfies $U_0 := \int_{-\infty} \bar{u}_0(x)\ dx \in H^{2,2}$, $||U_0||_{H^{2,2}} < \beta_0$, then the solution $u(x,t)$ to (1) with data $u(\cdot, 0) = \phi + \bar{u}_0$ exists for all times $t > 0$ and has

$$\lim_{t\to\infty} \sup_x |u(x,t) - \phi(x - st)| = 0$$

(i.e. ϕ is asymptotically stable).

The novelty consists in the absence of any convexity assumption on λ; thus the theorem generalizes a previous result by J. Goodman. In this note we only sketch the proof of this theorem and focus on motivating the choice of a new weight function that enables overcoming the technical difficulties caused by the absence of convexity.

1. Introduction

Consider a system of viscous conservation laws

$$u_t + f(u)_x = \mu u_{xx} \tag{1}$$

whose inviscid part $u_t + f(u)_x$ is hyperbolic, i.e. the Jacobian $f'(u)$ of the flux function $f : \mathbb{R}^n \to \mathbb{R}^n$ can be diagonalized over $\mathbb{R}$ at any state u. Assume that f' has a simple eigenvalue λ in a neighborhood of some reference state u_* and let ϕ denote the profile of a Laxian shock wave, i.e., $\phi : \mathbb{R} \to \mathbb{R}^n$ solves

$$\mu\phi' = h(\phi) := f(\phi) - s\phi - (f(u_-) - su_-) \quad , \quad \phi(\pm\infty) = u_\pm \ ,$$

where u_-, u_+, s satisfy $f(u_-) - su_- = f(u_+) - su_+$, $\lambda(u_-) > s > \lambda(u_+)$, and u_-, u_+ are close to u_*. The following is proved in [F1]:

Theorem 1.1. *There are numbers ϵ_0, $\beta_0(u_-, u_+) > 0$ such that whenever $|u_\pm - u_*| < \epsilon_0$ and $\bar{u}_0 \in L^1$ satisfies $U_0 := \int_{-\infty} \bar{u}_0(x)\ dx \in H^{2,2}$, $||U_0||_{H^{2,2}} < \beta_0$, then* *the solution $u(x,t)$ to (1) with data $u(\cdot,0) = \phi + \bar{u}_0$ exists for all times $t > 0$ and has*

$$\lim_{t\to\infty} \sup_x |u(x,t) - \phi(x-st)| = 0 \ .$$

The same result was previously known by Goodman [G1] under the additional assumption that the eigenvalue λ be convex, i.e. $r\cdot\nabla\lambda \neq 0$ where $\mathbb{R}r = \ker(f' - \lambda I)$. In other words the novelty of Theorem 1.1 consists in showing the asymptotic stability of traveling-wave profiles for small-amplitude shocks associated with possibly *non-convex* modes.

In this note we only sketch the proof of Theorem 1.1 and focus on motivating the choice of a new weight function for the primary field that enables overcoming the technical difficulties caused by the absence of convexity.

The Theorem is restricted to zero total mass, because $U_0 \in H^{2,2}$ implies $\int_{-\infty}^{\infty} \bar{u}_0(x)\, dx = 0$. A similar result can be obtained for non-zero mass perturbation, see [F2]. In its proof – using the approach of Szepessy and Xin, [SX] – the same weight function is essential in different ways.

2. Basic Steps

The proof of Theorem 1.1 consists of a short time existence result and an a-priori estimate from which global existence and the desired stability result follow. Focusing on the latter, we consider the difference between the solution $u^*(x,t) = \phi(x-st)$ and a solution u (corresponding to the perturbed initial data $\phi + \bar{u}_0$). Let $\bar{u}(x,t) := u(x,t) - \phi(x-st)$, then from (1)

$$\bar{u}_t + (f(u^* + \bar{u}) - f(u^*))_x = \mu \bar{u}_{xx} \ . \tag{2}$$

Multiplying (2) and $(2)_x$ by $\bar{u}$ and integrating $\int_{-\infty}^{\infty} dx$, $\int_{t_0}^{T} dt$ we easily arrive at

$$||\bar{u}(\cdot,T)||_{H^{1,2}} + \int_{t_0}^{T} ||\bar{u}_x(\cdot,t)||_{H^{1,2}}\ dt \leq C\big(||\bar{u}(\cdot,t_0)||_{H^{1,2}} + \int_{t_0}^{T} ||\bar{u}(\cdot,t)||_{L^2}\ dt\big). \tag{3}$$

Without the last term on the right hand side we would have an a-priori estimate for $||\bar{u}(\cdot,T)||_{H^{1,2}}$ and the decay of $||\bar{u}_x(\cdot,t)||_{H^{1,2}}$. To absorb the term on the r.h.s. and to get the decay of $||\bar{u}(\cdot,t)||_{H^{1,2}}$ (and hence of $\sup_x |\bar{u}|$) we thus consider the anti-derivative (or “integrated perturbation”)

$$U(x-st,t) = \int_{-\infty}^{x} \bar{u}(\xi,t)\ d\xi = \int_{-\infty}^{x} u(\xi,t) - \phi(\xi - st)\ d\xi \ .$$

We integrate equation (2) and pass to the moving coordinates $(x-st,t)$:

$$\begin{aligned} -sU_x + U_t + f(\phi + U_x) - f(\phi) &= \mu U_{xx} \\ U(x,0) = U_0(x) &= \int_{-\infty}^{x} u(\xi,0) - \phi(\xi)\ d\xi \ . \end{aligned}$$

By using Taylor expansion

$$f(\phi + U_x) - f(\phi) = f'(\phi)U_x - F(\phi, U_x)$$

we thus arrive at the so called *integrated equation*

$$U_t + h'(\phi)U_x - \mu U_{xx} = F(\phi, U_x) \tag{4}$$

$$U(\cdot, 0) = U_0 = \int_{-\infty}^{\cdot} u(x,0) - \phi(x)\ dx\ ,$$

where $h' = f' - s\mathrm{I}$. Together with (3) it is thus sufficient to show

$$||U(\cdot,T)||_{L^2}^2 + \int_{t_0}^{T} ||U_x(\cdot,t)||_{L^2}^2\ dt \leq C||U(\cdot,t_0)||_{L^2}^2\ , \tag{5}$$

which leads to

$$||U(\cdot,T)||_{H^{2,2}}^2 + \int_{t_0}^{T} ||U_x(\cdot,t)||_{H^{2,2}}^2\ dt \leq C||U(\cdot,t_0)||_{H^{2,2}}^2\ . \tag{6}$$

From the above, global existence can be obtained and thus, using (6) with $T = \infty$, we get the existence of a sequence (t_n) with $t_n \to \infty$ and

$$||U_x(\cdot,t_n)||_{L^2}^2 \to 0 \qquad , \qquad \int_{t_n}^{\infty} ||U_x(\cdot,t)||_{L_2}^2\ dt \to 0\ . \tag{7}$$

From (3) (with $t_0 = t_n$) and (7) we get

$$\begin{aligned}
&\lim_{t\to\infty} \sup_x |\phi(x - st) - u(x,t)|^2 = \lim_{t\to\infty} \sup_x |U_x(x,t)|^2 \\
&\leq \lim_{t\to\infty} ||U_{xx}(\cdot,t)||_{L^2}^2 \cdot ||U_x(\cdot,t)||_{L^2}^2 \leq \lim_{t\to\infty} C||U_x(\cdot,t)||_{L^2}^2 \\
&\overset{(3)}{\leq} C \lim_{t\to\infty} \left(||U_x(\cdot,t_n)||_{L^2}^2 + \int_{tn}^{t} ||U_x(\cdot,\tau)||_{L^2}^2\right) \to 0 \quad (t_n \to \infty)\ .
\end{aligned}$$

Thus we concentrate on the proof of (5) where a weight function is essential in the non-convex case.

3. Motivation of the weight function

Our choice of the weight function is inspired by the choice of Matsumura and Nishihara for the non-convex scalar case [MN]. Here we present the proof of (5) for the convex and non-convex scalar case to motivate the choice of our weight and to discuss some difficulties that arise in the case of a non-convex system.

For simplicity we assume $F \equiv 0$ in (4):

$$U_t + h'(\phi)U_x - \mu U_{xx} = 0\ . \tag{8}$$

3.1. The scalar case with convex flux

Multiplying (8) by U and integrating by parts $\int_{-\infty}^{\infty} dx$ we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} |U|^2 \, dx + \int_{-\infty}^{\infty} U h'(\phi) U_x + \mu (U_x)^2 \, dx = 0$$

and by integrating $\int_{t_0}^{T} dt$

$$\begin{aligned} \frac{1}{2}||U(\cdot,T)||_{L^2}^2 \quad &+ \quad \int_{t_0}^{T}\int_{-\infty}^{\infty} -\frac{1}{2}(h'(\phi))_x U^2 \, dx \, dt \\ &+ \quad \int_{t_0}^{T} ||U_x||_{L^2}^2 \, dt \ \le \ \frac{1}{2}||U(\cdot,t_0)||_{L^2}^2 \ . \end{aligned}$$

Together with the assumption of convexity $-(h'(\phi))_x > 0$, the inequality above implies (5).

3.2. The scalar case with non-convex flux

Multiplying (8) by $U \cdot w$, where $w = w(x)$, and integrating by parts $\int_{-\infty}^{\infty} dx$ we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} (w|U|^2) \, dx + \int_{-\infty}^{\infty} U w h'(\phi) U_x + \mu U w_x U_x + \mu w (U_x)^2 \, dx = 0$$

and by integrating $\int_{t_0}^{T} dt$

$$\begin{aligned} \frac{1}{2}||\sqrt{w}U(\cdot,T)||_{L^2}^2 \quad &+ \quad \int_{t_0}^{T}\int_{-\infty}^{\infty} -\frac{1}{2}(wh'(\phi)+w_x)_x U^2 \, dx \, dt \\ &+ \quad \int_{t_0}^{T} ||\sqrt{w}U_x||_{L^2}^2 \, dt \ \le \ \frac{1}{2}||\sqrt{w}U(\cdot,t_0)||_{L^2}^2 \ . \end{aligned}$$

Due to the presence of the weight w we get $-\frac{1}{2}(wh'(\phi)+w_x)_x$ in place of $-\frac{1}{2}(h'(\phi))_x$. Therefore the task is to find a positive w such that:

$$-\frac{1}{2}\left(wh'(\phi)+\mu w_x\right)_x > 0 \ .$$

Using the ansatz $w(x) = \tilde{w}(\phi(x))$ we get

$$\begin{aligned} -\frac{1}{2}\left(wh'(\phi)+\mu w_x\right)_x \quad &= \quad -\frac{1}{2}\left(\tilde{w}(\phi)h'(\phi)+\mu\tilde{w}'(\phi)\phi_x\right)_x \\ &= \quad -\frac{1}{2}\left((\tilde{w}h)'(\phi)\right)_x = -\frac{1}{2}(\tilde{w}h)''(\phi)\phi_x \end{aligned} \tag{9}$$

and by choosing

$$\tilde{w}(u) = -\frac{(u-u_+)(u-u_-)}{h(u)} \cdot \mathrm{sign}\phi_x > 0 \tag{10}$$

we obtain

$$-\frac{1}{2}\left(wh'(\phi)+\mu w_x\right)_x = |\phi_x| \ .$$

Obviously, in the case of a system (9) is not valid and (10) has no meaning, because then $h(u)$ is a vector.

4. Weight for the system case with non-convex mode

Like Goodman [G1] we diagonalize $h'(\phi)$. As f' (and hence h') is $\mathbb{R}$-diagonalizable in a neighborhood of the reference state u_* comprising the values of ϕ, we find smooth matrix valued functions

$$L(x) = \tilde{L}(\phi(x)), \quad R(x) = \tilde{R}(\phi(x))$$

such that $LR \equiv I$ and

$$L(h'(\phi))R = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \ ,$$

where

$$\lambda_p = \lambda - s \, , \qquad \lambda_i < 0 < \lambda_j \quad (i < p < j)$$

We substitute $U =: RV$ in (4), multiply by $V^T WL$ and integrate $\int_{-\infty}^{\infty} dx$, where $W = \mathrm{diag}(1, \ldots, 1, w, 1, \ldots, 1)$:

$$\int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial t}(V^T W V) + V^T W \Lambda V_x + V^T W \Lambda L R_x V$$
$$+\mu(V^T W L)_x (RV)_x - V^T W L F(\phi, (RV)_x) \ dx = 0 \ .$$

We group the terms above as

(A1) $\frac{1}{2}\frac{\partial}{\partial t}(V^T W V)$
(A2) $(w\lambda_p + \mu w_x) V_p (V_p)_x$
(A3) $\sum_{k \neq p} (-\frac{1}{2}(\lambda_k)_x + l_k (r_k)_x \lambda_k)(V_k)^2$
(A4) $\mu w((V_p)_x)^2 + \mu \sum_{k \neq p} ((V_k)_x)^2$
(B1) $\sum_{j \neq p} (w\lambda_p + \mu w_x) l_p (r_j)_x V_p V_j$
(B2) $\sum_{i \neq p, i \neq j} \lambda_i l_i (r_j)_x V_i V_j$
(B3) $\mu V^T W L_x R V_x + \mu V_x^T W L R_x V$
(B4) $\mu V^T W L_x R_x V$
(B5) $-V^T W L F(\phi, (RV)_x)$

where $L = (l_1, \ldots, l_n)^{\mathrm{T}}$ and $R = (r_1, \ldots, r_n)$.

The terms (B1)–(B5) have to be estimated; they mainly consist of coupling terms. (A1)–(A4) are what we expect from a decoupled system. In particular (A2) – the term referring to the primary field – corresponds to the scalar model case of Section 3.2. Our first task is thus to “find” a positive weight w such that

$$-\frac{1}{2}(w\lambda_p + \mu w_x) = |\phi_x| \ .$$

For this we can prove

Lemma 4.1. *For* $\epsilon := |u_- - u_+|$ *sufficiently small there exists* $w : \mathbb{R} \to \mathbb{R}$ *with* $\inf_x w(x),\ \inf_x(1/w(x)) > 0$ *and*

$$-\frac{1}{2}(w\lambda_p + \mu w_x)_x = |\phi_x| \tag{11}$$

Proof. For fixed x_0, w_0 let w be the solution of

$$\begin{aligned} \mu w_x + w\lambda_p &= -2\int_{x_0}^{x} |\phi_x|\; dx \\ w(0) &= w_0, \end{aligned} \tag{12}$$

i. e.,

$$w(x) = e^{-\int_0^x \frac{\lambda_p(\xi)}{\mu}\, d\xi} \left(w_0 - \int_0^x e^{\int_0^y \frac{\lambda_p(\xi)}{\mu}\, d\xi} \frac{2}{\mu} \left[\int_{x_0}^{y} |\phi_x|\; d\xi \right] dy \right) . \tag{13}$$

We now choose x_0 such that

$$I(x_0) \equiv \int_{-\infty}^{\infty} e^{\int_0^y \frac{\lambda_p(\xi)}{\mu}\, d\xi} \frac{2}{\mu} \left[\int_{x_0}^{y} |\phi_x|\; d\xi \right] dy \ = \ 0 \ ,$$

and correspondingly

$$w_0 = \int_0^{\pm\infty} e^{\int_0^y \frac{\lambda_p(\xi)}{\mu}\, d\xi} \frac{2}{\mu} \left[\int_{x_0}^{y} |\phi_x|\; d\xi \right] dy, \ .$$

By $\lambda_p(-\infty) > 0 > \lambda_p(\infty)$ the function I is well defined and $I(-\infty) > 0 > I(\infty)$ implies the existence of x_0. This choice is equivalent to the boundedness of w. By (13)

$$\lim_{x\to\pm\infty} e^{\int_0^x \frac{\lambda_p}{\mu}\, d\xi} \cdot w(x) = 0 \ ,$$

by (12)

$$\left(e^{\int_0^x \frac{\lambda_p}{\mu}\, d\xi} \cdot w\right)' = \frac{-2}{\mu} e^{\int_0^x \frac{\lambda_p}{\mu}\, d\xi} \int_{x_0}^{x} |\phi_x|\; dx \begin{cases} > \ 0 & \text{for } x < x_0 \\ = \ 0 & \text{for } x = x_0 \\ < \ 0 & \text{for } x > x_0 \ . \end{cases}$$

Consequently $e^{\int_0 \frac{\lambda_p}{\mu}\, d\xi} \cdot w > 0$ and thus $w > 0$.
As $w(\pm\infty) = -2\int_{x_0}^{\pm\infty} |\phi_x| dx/\lambda_p(\pm\infty) > 0$ by (12), both $\inf w$ and $\inf(1/w)$ are positive. □

We now turn to the discussion of the coupling terms. Guided by the explicit choice (10) which was used for the non-convex scalar case we see that if the flux $h(\phi)$ is of the order ϵ^{-3} (a natural thing in the non-convex case) then the weight w is of the order ϵ^{-1}. On the other hand – for a system ϵ has to be chosen small to control the coupling between the different fields. Although this seems to cause difficulties, taking a closer look to our grouping of the coupling terms, we see that either w appears as $w\lambda_p + \mu w_x$ or as $w\phi_x$ where both can be estimated to be of order ϵ or ϵ^2. In the scalar case of Section 3.2 this can be explicitly verified. For (11) we can prove

Lemma 4.2. *For the weight function w of Lemma 4.1 the following holds:*

$$|w\lambda_p + \mu w_x| \leq O(1)|u_+ - u_-| \tag{14}$$

$$|\mu(w\phi_x)_x| = |(wh(\phi))_x| \leq O(1)|u_+ - u_-| \cdot |\phi_x| \tag{15}$$

$$|\mu w\phi_x| = |wh(\phi)| \leq O(1)|u_+ - u_-|^2 \tag{16}$$

For the proof we refer to [F1].

By Lemma 4.2 the proof of (5) can be completed. E.g. (11) gives positivity of (A2) and (14), (15) and (16) enable estimating (B1), (B3) and (B4). With (A3) and (B2) we deal as suggested in [G2] for the case of a convex system.

References

[F1] Fries, C., *Nonlinear asymptotic stability of general small-amplitude viscous Laxian shock waves*, J. Differ. Equations, to appear.

[F2] Fries, C., Doctoral Thesis, in preparation.

[G1] Goodman, J., *Nonlinear Asymptotic Stability of Viscous Shock Profiles for Conservation Laws*, Arch. Rational Mech. Anal., **95**, (1986), 325-344.

[G2] Goodman, J., *Remarks on the Stability of Viscous Shock Waves*, Viscous Profiles and Numerical Methods for Shock Waves, Ed. M. Shearer, SIAM, Philadelphia, 1991, 66-72.

[MN] Matsumura, A., Nishihara K., *Asymptotic Stability of Traveling Waves for Scalar Viscous Conservation Laws with Non-convex Nonlinearity*, Commun. Math. Phys., **165**, (1994), 83-96.

[SX] Szepessy, A., Xin, Z., *Nonlinear Stability of Viscous Shock Waves,* Arch. Rational Mech. Anal., **122**, (1993), 53-103.

Institut für Mathematik
RWTH Aachen
D 52072 Aachen, Germany
E-mail address: fries@instmath.rwth-aaachen.de