Time-asymptotic stability of shock profiles in the presence of diffusion waves

1. Viscous Shock Profiles

Hyperbolic conservation law

$$u_t + f(u)_x = \mu u_{xx} \quad x \in \mathbb{R},\ u \in \mathbb{R}^n \qquad (1)$$
$$u(\pm\infty) = u_\pm$$

(i.e. f' has only real eigenvalues).

Consider a traveling wave solution (profile of a Laxian shock wave)

$$u(x,t) = \phi(x - st)$$

associated with the simple eigenvalue λ of f', i.e., ϕ solves

$$\mu\phi' = h(\phi) = f(\phi) - s\phi - q \quad ,$$
$$\phi(\pm\infty) = u_\pm \ ,$$

where u_-, u_+, s, q satisfy

$$f(u_-) - su_- = f(u_+) - su_+ = q \quad ,$$
$$\lambda(u_-) > s > \lambda(u_+)$$

Goal:

Theorem 1
There is a positive constant ϵ_0 *such that if* u_-, u_+ *satisfy* $|u_\pm - u_*| < \epsilon_0$, *then there exists* $\beta_0 > 0$ *(depending on* f, u_-, u_+ *and* μ*) such that whenever the perturbation* $u_0 - \phi \in \mathrm{H}^1(\mathbb{R})$ *satisfies*

$$\int_{-\infty}^{\infty} |u_0(x) - \phi(x - \delta_0)|\, dx + \int_{-\infty}^{\infty} (1 + (x - \delta_0)^2)|u_0(x) - \phi(x - \delta_0)|^2\, dx \leq \beta_0$$

for some $\delta_0 \in \mathbb{R}$, *then the solution* $u(x,t)$ *to* (1) *with data* $u(\cdot, 0) = u_0$ *exists for all times* $t > 0$ *and has*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi(x - st - \delta)| = 0 \qquad (2)$$

with a uniquely determined $\delta \in \mathbb{R}$.

Note:

- No convexity assumption.
- Non-zero mass perturbations.

Sketch of proof

1. Diagonalization

Change to moving coordinates $x \to x - st$:

$$u_t + (h(u))_x = \mu u_{xx}$$

where $h' = f' - s\mathrm{I}$.

$\exists\, L = (l_i)$, $R = (r_j)$ matrix valued functions such that

$$L(u)h'(u)R(u) = \Lambda(u) = \mathrm{diag}(\underbrace{\lambda_1, \ldots,}_{<0} \lambda_p, \underbrace{\ldots, \lambda_n}_{>0})$$

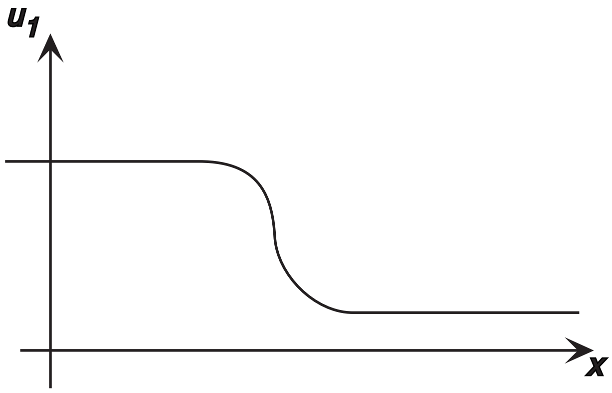
$$\lambda_p(u_-) > 0 > \lambda_p(u_+)$$

2. Decomposition of mass

Let δ, $\theta_{i,0}$ ($1 \leq i \leq n$, $i \neq p$) be defined by

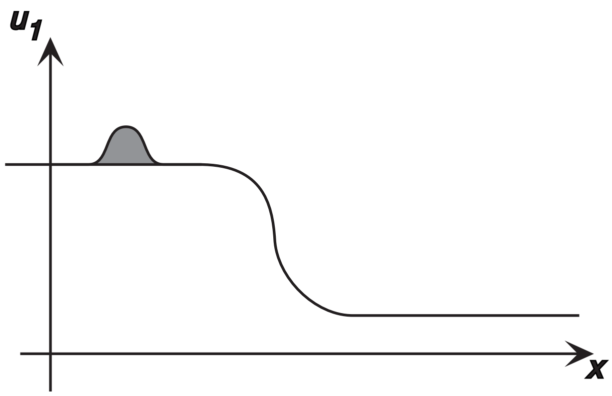
$$\begin{aligned}&\text{“mass of perturbation”}\\ &= \int_{-\infty}^{\infty} u_0(x) - \phi(x)\, dx\\ &=: \ \delta(u_+ - u_-) + \sum_{i \neq p} \theta_{i,0} r_i(u_{i,0})\end{aligned}$$

where $u_{i,0} := u_{\mathrm{sign}(i-p)} \in \{u_-, u_+\}$.

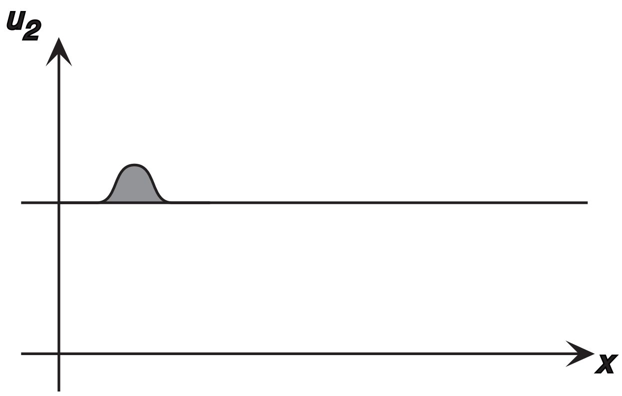


Shock profile (viewed from a side).

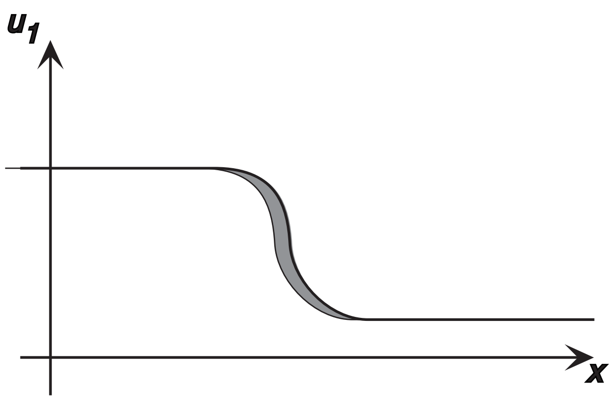
Shock profile (viewed from another side)



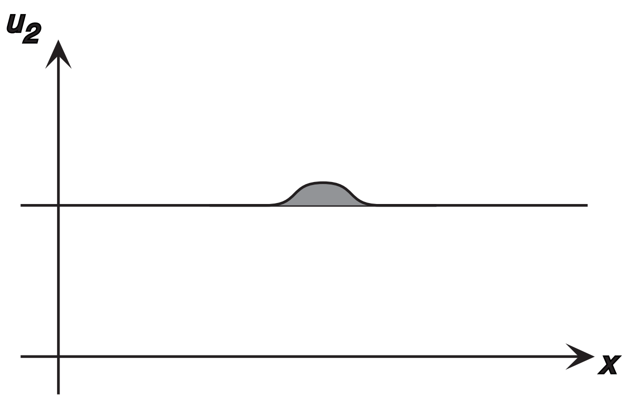
Perturbation with non-zero mass...



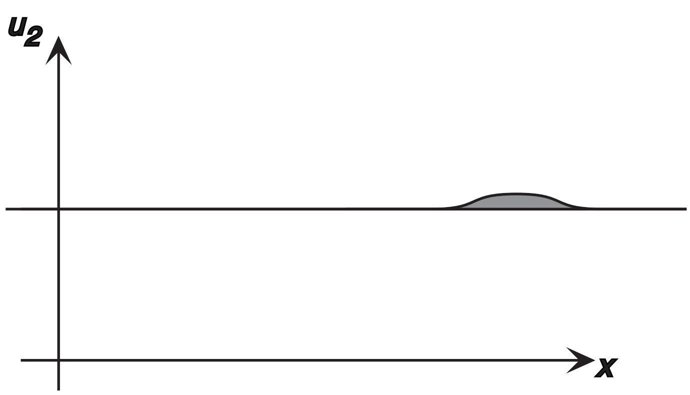
Perturbation with non-zero mass...



...leads to a shift of the wave.



...evolves like a diffusion wave:



3. Define decoupled diffusion wave θ by

$$\theta := \sum_{i \neq p} \theta_i r_i(u_{i,0})$$

$$\theta_i(x,t) = \frac{\theta_{i,0}}{\sqrt{1+t}} v_i \left(\frac{(x-x_0) - \lambda_i(u_{i_0})(1+t)}{2\sqrt{1+t}} \right)$$

as solution of

$$\theta_{i,t} + (\lambda_i(u_{i_0})\theta_i + \frac{1}{2}\nabla\lambda_i(u_{i_0}) \cdot r_i(u_{i,0})\theta_i^2)_x - \mu\theta_{i,xx} = 0$$

$$\int_{-\infty}^{\infty} \theta_i(x,t)dx = \theta_{i,0}$$

The solution of this equation is known explicitly!
It decays as $\frac{O(1)}{\sqrt{1+t}}$.

4. Define the coupled linear diffusion wave η by

$$\eta_t + (h'(\phi)\eta)_x - \mu\eta_{xx} = E_{1,x} + E_{2,x}$$

$$\eta(x,0) \equiv 0$$

where E_1 is a θ-θ coupling term and E_2 is a θ-η coupling term (both bilinear).

5. Decompose solution as

$$u = \phi(\cdot - \delta) + \theta + \eta + w$$

i.e.

$$w := u - (\phi(\cdot - \delta) + \theta + \eta)$$

To do:

- Pointwise estimate for θ.
  (Easy: $|\theta(\cdot, t)| \leq \frac{O(1)}{\sqrt{1+t}}$ since decoupled diffusion wave is known explicitly).
- Pointwise estimate for η.
  (Will look like $|\eta(\cdot, t)| \leq \frac{\log(1+t)}{\sqrt{1+t}}$)
- Global existence for w and
  $\sup_x |w(x,t)| \to 0 \ (t \to \infty)$

Implies:

Global existence of u and

$$\sup_x |u - \phi(\cdot - \delta)| \to 0 \ \ (t \to \infty).$$

[No decay rate, because energy method is used for w].

6. Pointwise estimate for η – sketch of proof:

$$\eta_t + (h'(\phi)\eta)_x - \mu\eta_{xx} = E_{1,x} + E_{2,x}$$
$$\eta(x,0) \equiv 0$$

a) Integrate and diagonalize:

$$d(x,t) := L(\phi)\int_{-\infty}^{x} \eta(\xi,t)dx$$

$$d_t + \Lambda(\phi)d_x - \mu d_{xx} = LE_1 + LE_2 - \Lambda L R_x d$$
$$- 2\mu L R_x d_x + \mu L R_{xx} d$$

Intermediate goal: $|d(x',T)| \leq \frac{1+\log(1+T)}{(1+T)^{1/2}}$

b) Define (approximate) Green's functions:

$$-\psi_{i,t} - (\lambda_i\psi_i)_x - \mu\psi_{i,xx} = 0$$
$$\psi_i(x',T) = \delta(x'-x)$$

(*dual wave*). Leads to

$$d_i(x',T) = \int_0^T \int_{-\infty}^{\infty} \psi_i[LE_1 + LE_2 - \Lambda L R_x d$$
$$+ 2\mu L R_x d_x + \mu L R_{xx} d]_i dxdt$$

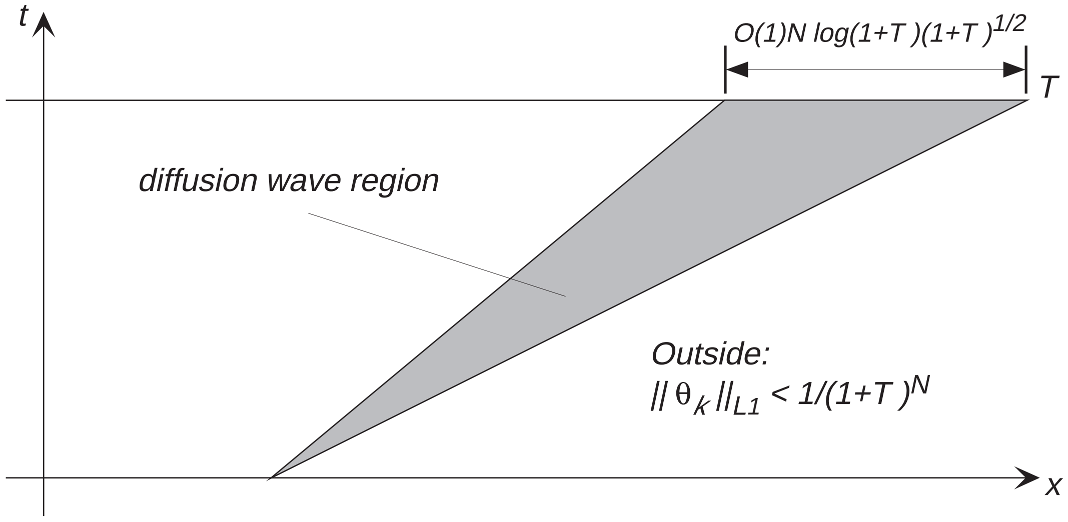
Example:

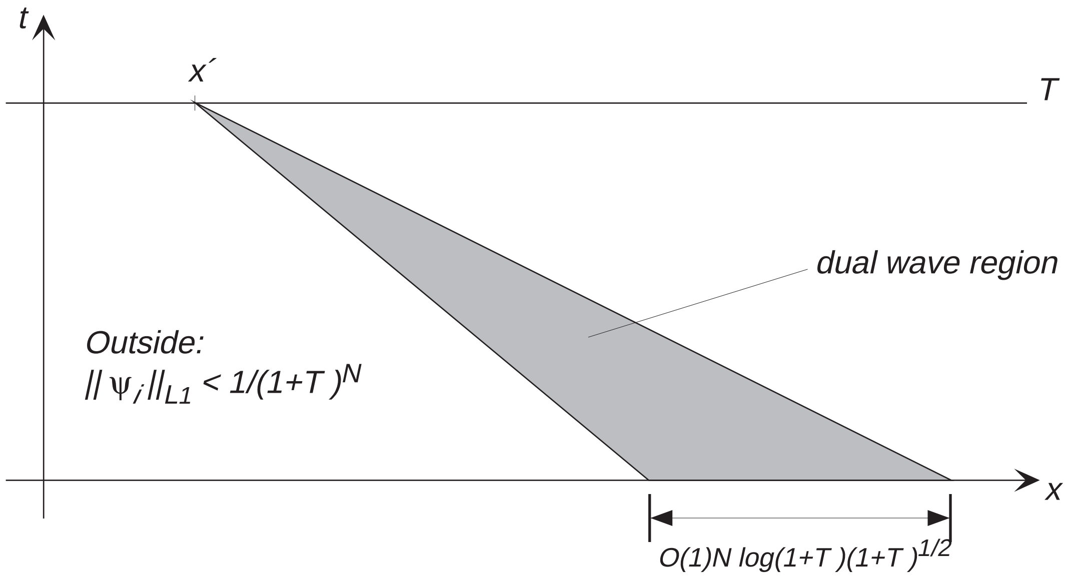
How to estimate

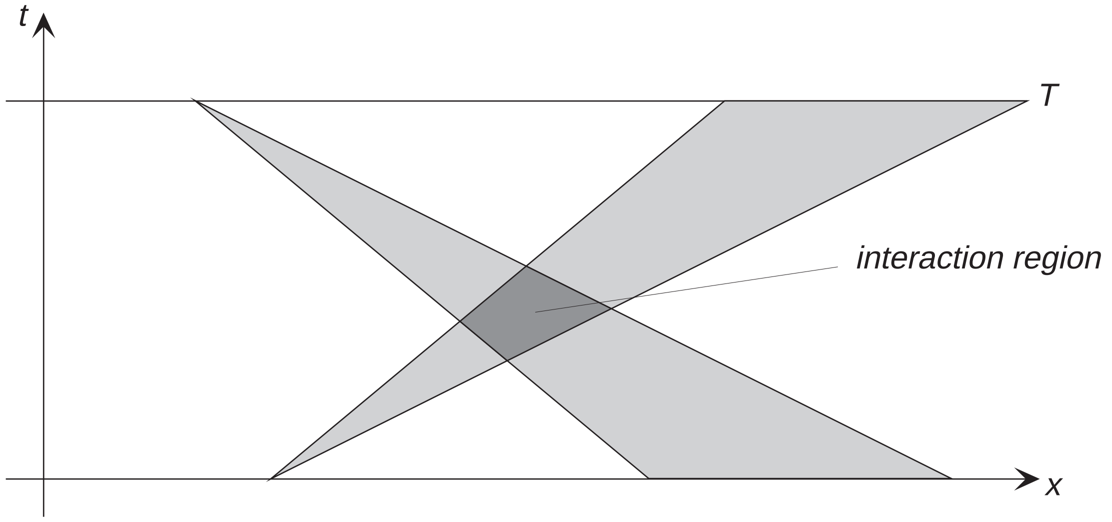
$$\int_0^T \int_{-\infty}^{\infty} |\psi_i(x,t)||\theta_k(x,t)|^2 dx dt \quad (i \neq k)$$

We need:

- The dual wave ψ_i decays backward in time.
- The dual wave ψ_i is localized (i.e. small outside a certain region).
- The diffusion wave decays and is localized


t
O(1)N log(1+T)(1+T)1/2
T
diffusion wave region
Outside:
|| θk ||L1 < 1/(1+T)N
x


t
x´
T
dual wave region
Outside:
|| ψi ||L1 < 1/(1+T)N
x
O(1)N log(1+T)(1+T)1/2


t
T
interaction region
x

Thus we can focus on the estimate of

$$\int_{\tau_1}^{\tau_2} \int_{-\infty}^{\infty} |\psi_i(x,t)||\theta_k(x,t)|^2 dxdt \quad (i \neq k)$$

where "interaction-region" $\subset \mathbb{R} \times [\tau_1, \tau_2]$.

a) In the case of "early" interaction ($\tau_2 \leq \frac{T}{2}$) use the backward decay of ψ_i:

$$\ldots \leq ||\psi_i(\cdot,\tau_2)||_{\mathsf{L}_\infty} \int_{\tau_1}^{\tau_2} ||\theta_k^2(x,t)||_{\mathsf{L}_1} dt$$

b) In the case of "late" interaction ($\tau_1 \geq \frac{T}{2}$) use the decay of θ_k

$$\ldots \leq ||\theta_k^2(\cdot,\tau_1)||_{\mathsf{L}_\infty} \int_{\tau_1}^{\tau_2} ||\psi_i(\cdot,t)||_{\mathsf{L}_1} dt$$

and obtain

$$\ldots \leq \frac{\mathsf{O}(1)}{(1+T)^{1/2}} \log(1+T)$$

Example:

How to estimate

$$\int_0^T \int_{-\infty}^{\infty} |\psi_i(x,t)||\phi_x(x)|^2 |d(x,t)| dx dt$$

Let $T_1 \leq T \leq T_1 + 1$. Because of

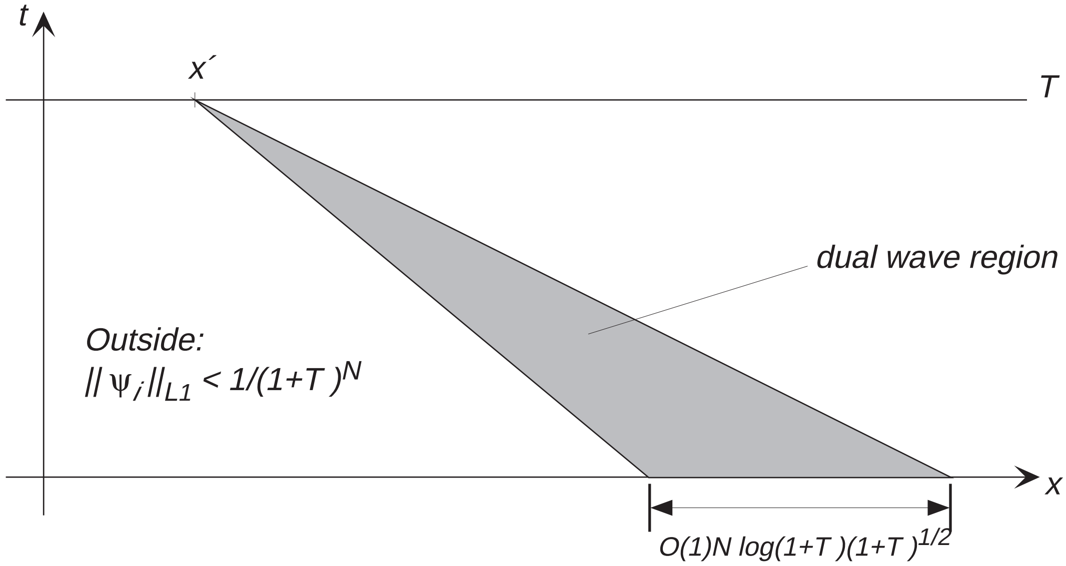
$$\int_0^T \int_{-\infty}^{\infty} \ldots dx dt$$
$$\leq \int_0^{T_1} \int_{-\infty}^{\infty} \ldots dx dt + \epsilon_0 \sup_{T_1 \leq t \leq T} ||d(\cdot,t)||$$

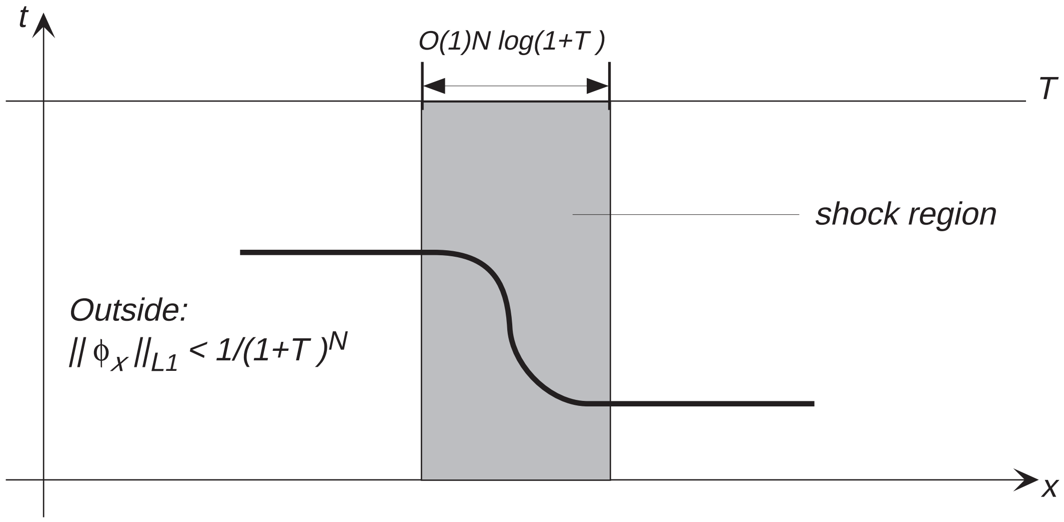
we can focus on the integral over $[0, T_1]$, where we use the induction assumption

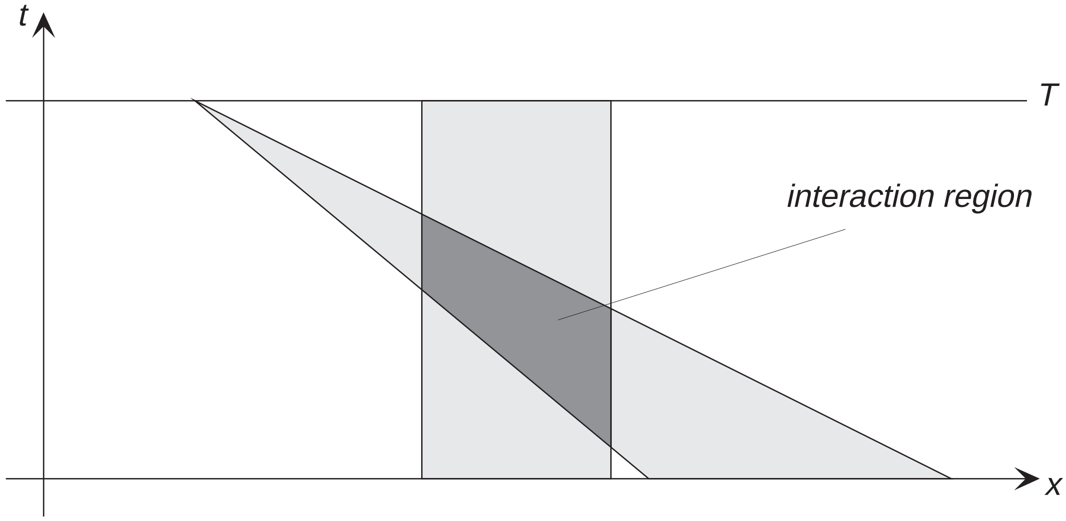
$$||d(\cdot,t)||_{L_\infty} \leq \frac{1 + \log(1+t)}{\sqrt{1+t}} \quad \forall\, 0 < t \leq T_1$$

We need:

- The dual wave ψ_i decays backward in time.
- The dual wave ψ_i is localized (i.e. small outside a certain region.
- ϕ_x is small outside a certain region.


t
x´
T
dual wave region
Outside:
$\|\psi_i\|_{L1} < 1/(1+T)^N$
x
O(1)N log(1+T)(1+T)$^{1/2}$


t
O(1)N log(1+T)
T
shock region
Outside:
$\|\phi_x\|_{L1} < 1/(1+T)^N$
x


t
T
interaction region
x

Thus we can focus on the estimate of

$$\iint_{\substack{\text{interaction}\\\text{-region}}} |\psi_i(x,t)||\phi_x(x)|^2|d(x,t)|dxdt$$

where we have: “interaction-region” $\subset \mathbb{R} \times [\tau_1, \tau_2]$.

a) In the case of “early” interaction ($\tau_2 \leq \frac{T}{2}$) use

$$\leq K||\psi_i(\cdot,\tau_2)||_{\mathsf{L}_\infty}||\phi_x||_{\mathsf{L}_1}$$

and the (backward in time) decay of ψ_i.

b) In the case of “late” interaction ($\tau_1 \geq \frac{T}{2}$) use

$$\leq ||d(\cdot,\tau_1)||_{\mathsf{L}_\infty} \int_{-\infty}^{\infty}\int_{\tau_1}^{\tau_2} |\psi_i(x,t)|dt|\phi_x|^2 dx$$

and the induction assumption on d. It remains to show:

Lemma (vertical estimate)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\tau_2} |\psi_i(x,t)|dt|\phi_x|^2 dx \leq K\epsilon_0$$

It is non-trivial that this lemma holds in the non-convex case.