

A recursive formula for the Kurtosis
of an approximation to the distribution of share prices

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Appendix: A recursive formula for the Kurtosis of an approximation to the distribution of share prices

Introduction

Given the recursive approximation to the distribution of share prices (see [1, 2]) we will derive a recursive formula for Kurtosis. The Kurtosis of a distribution consisting of samples $x_1 \dots x_N$ with mean μ and standard deviation σ is given by

$$K = \left(\frac{N \cdot (N+1)}{(N-1) \cdot (N-2) \cdot (N-3)} \sum_{i=1}^N \left(\frac{x_i - \mu}{\sigma} \right)^4 \right) - \frac{3(N-1)^3}{(N-2) \cdot (N-3)}. \quad (1)$$

Recursive calculation of Kurtosis

We fix notation as follows: Assuming that a distribution of prices p_j ($j = 1, \dots, n$) occurring at a volume v_j is given (say at time t_n). At later time (say at time t_{n+1}) the distribution changed to

- one price p_{n+1} occurring at volume v_{n+1}
- the prices p_1, \dots, p_n occurring at volumes $\frac{N-v_{n+1}}{N}v_1, \dots, \frac{N-v_{n+1}}{N}v_n$, respectively

(where $N = \sum_{j=1}^n v_j = \sum_{j=1}^n \frac{N-v_{n+1}}{N}v_j + v_{n+1}$ is the total volume of shares floating).

The Kurtosis of the distribution at time t_n is thus given by

$$K_n = \left(\frac{N \cdot (N+1)}{(N-1) \cdot (N-2) \cdot (N-3)} \sum_{i=1}^n v_i \left(\frac{p_i - \mu_n}{\sigma_n} \right)^4 \right) - \frac{3(N-1)^3}{(N-2) \cdot (N-3)}$$

and the Kurtosis of the distribution at time t_{n+1} is given by

$$K_{n+1} = \left(\frac{N \cdot (N+1)}{(N-1) \cdot (N-2) \cdot (N-3)} \sum_{i=1}^{n+1} \tilde{v}_i \left(\frac{p_i - \mu_{n+1}}{\sigma_{n+1}} \right)^4 \right) - \frac{3(N-1)^3}{(N-2) \cdot (N-3)},$$

where $\tilde{v}_i = \frac{N-v_{n+1}}{N} v_i$ for $i = 1, \dots, n$ and $\tilde{v}_{n+1} = v_{n+1}$.

To have an algorithm that allows calculation of Kurtosis in an recursive way (more precisely: to give Kurtosis as a function of recursively defined values) let us define

$$\begin{aligned} q_n^{(1)} &= \sum_{i=1}^n v_i p_i, & q_{n+1}^{(1)} &= \frac{N-v_{n+1}}{N} q_n^{(1)} + v_{n+1} p_{n+1} \\ q_n^{(2)} &= \sum_{i=1}^n v_i p_i^2, & q_{n+1}^{(2)} &= \frac{N-v_{n+1}}{N} q_n^{(2)} + v_{n+1} p_{n+1}^2 \\ q_n^{(3)} &= \sum_{i=1}^n v_i p_i^3, & q_{n+1}^{(3)} &= \frac{N-v_{n+1}}{N} q_n^{(3)} + v_{n+1} p_{n+1}^3 \\ q_n^{(4)} &= \sum_{i=1}^n v_i p_i^4, & q_{n+1}^{(4)} &= \frac{N-v_{n+1}}{N} q_n^{(4)} + v_{n+1} p_{n+1}^4, \end{aligned}$$

i.e.

$$q_{n+1}^{(j)} := \frac{N-v_{n+1}}{N} q_n^{(j)} + v_{n+1} p_{n+1}^j = \left(\frac{N-v_{n+1}}{N} \sum_{i=1}^n v_i p_i^j \right) + v_{n+1} p_{n+1}^j. \quad (2)$$

With these definitions we have

$$\sum_{i=1}^n v_i \left(\frac{p_i - \mu_n}{\sigma_n} \right)^4 = \frac{1}{\sigma_n^4} \left(q_n^{(4)} - 4\mu_n q_n^{(3)} + 6\mu_n^2 q_n^{(2)} - 4\mu_n^3 q_n^{(1)} + \mu_n^4 \right)$$

and

$$\begin{aligned} & \sum_{i=1}^n \frac{N-v_{n+1}}{N} v_i \left(\frac{p_i - \mu_{n+1}}{\sigma_{n+1}} \right)^4 + v_{n+1} \left(\frac{p_{n+1} - \mu_{n+1}}{\sigma_{n+1}} \right)^4 \\ &= \frac{1}{\sigma_{n+1}^4} \left(q_{n+1}^{(4)} - 4\mu_{n+1} q_{n+1}^{(3)} + 6\mu_{n+1}^2 q_{n+1}^{(2)} - 4\mu_{n+1}^3 q_{n+1}^{(1)} + \mu_{n+1}^4 \right) \end{aligned}$$

and thus we find that the Kurtosis of the distribution at any time t_m is given by

$$K_m = \frac{N \cdot (N+1)}{(N-1) \cdot (N-2) \cdot (N-3)} k \left(\mu_m, \sigma_m, q_m^{(1)}, q_m^{(2)}, q_m^{(3)}, q_m^{(4)} \right) - \frac{3(N-1)^3}{(N-2) \cdot (N-3)} \quad (3)$$

where μ_m and σ_m are mean and standard deviation respectively (which could be calculated through a recursive formula)¹, $q_m^{(1)}, \dots, q_m^{(4)}$ are defined by the recursion formulas above and

$$k \left(\mu, \sigma, q^{(1)}, q^{(2)}, q^{(3)}, q^{(4)} \right) := \frac{1}{\sigma^4} \left(q^{(4)} - 4\mu q^{(3)} + 6\mu^2 q^{(2)} - 4\mu^3 q^{(1)} + \mu^4 \right). \quad (4)$$

The initial values for the recursively defined $q_m^{(j)}$ are given by $q_0^{(j)} = N \cdot p_0^j$.

¹In fact, $\mu_m = \frac{1}{N} q_m^{(1)}$ and $\sigma_m = \sqrt{\frac{1}{N} (q_m^{(2)} - 2\mu_m q_m^{(1)} + \mu_m^2)}$.

References

- [1] FRIES, CHRISTIAN P.: Elastic Moving Averages, Technical Analysis of Stocks & Commodities, June 2001.
- [2] <http://www.ChristianFries.com/evwma/>

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