The Distribution of Share Prices
and
Elastic Time and Volume Weighted Moving Averages

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1 Introduction

In this article we consider "the distribution of prices paid per share" and derive simple approximation formulas for its mean and other statistical characteristica. The formulas we derive are approximations since the data about the distribution of prices paid by each shareholder for each share is not available. After going through the derivation of the formula we will end up with recursive definitions of time series which could be viewed as *random coefficient autoregressive (RCA) time series* (see [6, 5]).

At this stage, we have no other motivation for the derivation of such formulas than the mere hope that they should supply some useful information (e.g. whether the average shareholder paid for his shares less or more than the current value). Indeed, it turns out that the (approximation to) the mean of the distribution of prices paid per share is (as one might have expected) a smoothing of the time series of share prices.

In Section 4.1 we will thus re-define it as "moving average" and compare it to the standard moving averages used to smooth the time series of share prices.

However, it is not our intention to introduce yet another smoothing or moving average with yet another set of advantages (and of course disadvantages). Instead, we believe that the value of this article lies in the derivation given in Section 2 of the article, i.e. viewing the moving averages defined later as mean of distributions which model (approximate) the distribution of share prices and time value, respectively. For example the well known Exponential Weighted Moving Average (EWMA) will come out as a special case of our approximation for stocks with non-volatile volume.

A crucial difference of the method presented to the standard moving averages is the weightening by volume and the way it is done. To emphasize this we will conclude this introduction by giving two motivations.

Although we will illustrate the smoothing defined by the volume weighted moving average in Section 4.1 by applying it to stock price time series, it should be noted and emphasized that applications of such weightening schemes are wide and a similar approach might be useful in a variety of areas. To name a few areas where a volume weightening scheme like the one we present might be applied:

- Exponentially weighted moving averages (EWMA) and the (generalized) autoregressive conditional heteroscedasticity (ARCH, GARCH) model are used to estimate/model volatilities (see, e.g., [1, 3] or [4], Ch. 15), e.g. in J.P. Morgans "RiskMetrics" VAR Methodology where an EWMA approach is used.

- In portfolio theory, asset allocation models like e.g. the Black-Literman model try to find a weighting scheme which maximize the expected excess return over the risk-free rate while ensuring that there are no dramatic changes. To do so, the expected return is estimated by various estimators (e.g. a simple moving average or a regression model). A volume weighting scheme could be introduced here by altering the definition of "return" or introducing a volume weighting in the regression model.

- Some exotic derivative products explicitly include averages of stock prices in their definition (e.g. Asian options). In this context, it should be noted that splitting one big trade into two smaller ones would generally change a non-volume weighted average since the price enters twice. The volume weightening scheme presented here is invariant to such a splitting.

1.1 A motivation (I): the non-sense of disregarding volume

Standard moving average
First consider a three-day moving average (we consider only three days to keep the example simple). In addition assume that there is only one trade each day (again, this is only to keep the example simple).

Example 1		
day/trade no.	price	volume
1	$10	100
2	$20	200
3	$30	300

Example 2		
day/trade no.	price	volume
1	$10	300
2	$20	200
3	$30	100

A standard three-day moving average is constructed by summing up the prices of the last three days and dividing by the number of days (here three). In both examples the three-day moving average is $20. The price of $10 at day one is balanced by the price of $30 at day three. However, volume suggests that in the first example the first day should be weighted less and the third day should be weighted more and vice versa in the second example.

Standard volume weighted moving average
The above leads to the insight that the averaging should be volume weighted. A standard three-day volume weighted moving average is constructed by summing up the prices of the last three days, each multiplied by the respective volume and divided by

the total volume over the three days. The three-day volume weighted moving average in Example 1 is $\$23\frac{1}{3}$. The three-day volume weighted moving average in Example 2 is $\$16\frac{2}{3}$.

Example 3			
day/trade no.	price	volume	three-day volume weighted moving average
1	$30	280	N/A
2	$40	310	N/A
3	$30	30	$35
4	$20	50	$\$36\frac{2}{3}$
5	$35	40	$\$27\frac{1}{2}$

Example 3 presents the effect of the main defect of the three-day volume weighted moving average: information prior to the three days is not considered, even if volume suggests its importance[1]. In Example 3, the three-day volume weighted moving average of the first three days is $35. Although on day four we see a lower price of $20, the three-day volume weighted moving average (now considering only day 2, 3 and 4) rises. The reason for this behavior is that we no longer consider day 1, and (considering only day 2, 3 and 4) the high price at high volume on day 2 is becoming even more important as we "forget" day 1.

At day five the price rises to $35 (more or less the average value), but the three-day volume weighted moving average drops to $\$27\frac{1}{2}$. This is because on day 5 the averaging completely ignores day 1 and day 2 and considers only day 3, 4 and 5 among which day 4 is weighted most. We forget about day 1 and 2, although volume suggests that they give the most reliable prices.

1.2 A motivation (II): Does today's price contain all information and what is the definition of *return*

A frequent argument against so called "indicators" is that *today's price contains all information.* The argument stems from the assumption of no-arbitrage in complete markets. Although the author is very skeptic about the theoretical corroboration and practical meaning of almost all "indicators", the same skepticism should apply to this argument. Consider the following example:

Example 4		
trade no.	price	volume
1	$48	500
2	$50	500
3	$30	30
4	$35	20

Why should the last price be relevant, if the stock has been traded at a stable

[1]In fact, any moving average with a fixed time-frame has this problem

(average) price of \$49 at high volume? If for example the investor would like to buy 500 shares it might be that he would get 50 shares for an price around \$35 (maybe from the 50 people who bought the share approximately at that price), but due to the majority having paid prices around \$50 he might have to pay more for the remaining 450 shares.

In other words, disregarding volume (and following the above argument about today's price) means disregarding liquidity risk.

In this context one could also reconsider the definition of return. The (potential) return from trade 1 to trade 2 is $\$2/\$50 = 1\%$. The (potential) return from trade 3 to trade 4 is $\$5/\$35 \approx 14\%$. However, volume suggest that the return from trade 1 to 2 could be realized on 20 times more shares than the return from trade 3 to 4.

2 The distribution of prices paid per share

We assume that we would know for every share the price its current shareholder had paid for it. This would give us the distribution of prices paid per share.

This distribution and measures like its mean, variance or Kurtosis are generally not available. In the following we will model the distribution of prices paid per share and derive simple recursive approximation formulas for the above mentioned statistical measures. The approximation formula for the mean could then be tweaked to supply a smoothing similar (or advantageous) to moving averages. As a first try, let us find an approximation for the mean of the distribution of prices paid per share:

2.1 An approximation of the mean of the distribution of share prices

Let us assume that at its IPO N stocks are issued for a price p_0. In this case (lets say at time $t = 0$) the distribution of prices is known and the mean is just

$$\mu_0 := p_0. \tag{1}$$

Now assume that we would know the distribution (or at least its mean) at some time t_i and a shareholder would decide to sell n_i shares for a price of p_i and that the shareholder had initially paid a price of q_i for the shares he sells. Then the mean of the distribution changes to

$$\mu_i := \frac{1}{N}\left(N\mu_{i-1} + n_i(p_i - q_i)\right), \tag{2}$$

where N is the total number of shares floating and μ_{i-1} denotes the mean before the transaction took place[2].

Since q_i, i.e. the price the seller initially had paid for the stock he sells, is not known the above formula is unusable. But since we know that the average price that has been paid for shares at time t_{i-1} is μ_{i-1}, we might use μ_{i-1} as proxy for q_i and thus introduce $\tilde{\mu}_i$ as

[2]The formula (2) could be interpreted as follows: From the pool of N shares for which an average price of μ_{i-1} and thus a total price of $N\mu_{i-1}$ has been paid, n_i shares for which a price q_i has been paid are removed and n_i shares for which a price p_i has been paid are added.

$$
\begin{aligned}
\tilde{\mu}_0 &:= p_0 \\
\tilde{\mu}_i &:= \frac{1}{N}\left(N\tilde{\mu}_{i-1} + n_i(p_i - \tilde{\mu}_{i-1})\right) \\
&= \frac{1}{N}\left((N - n_i)\tilde{\mu}_{i-1} + n_i p_i\right),
\end{aligned}
\tag{3}
$$

The resulting recursive definition (3) defines a *random coefficient autoregressive time series model* (see e.g. [6]), where the underlying time series is given by the stock price process p_i and the random coefficients are derived from the volume process n_i. We will make a few comments on random coefficient autoregressive time series in Section 3.

We would like to conclude our discussion of the distribution of share prices by deriving a simple model (i.e. approximation) for the evolution of the distribution itself, which enables us to find similar recursive formulas for approximations of other statistical measures, e.g. the Standard Deviation, Skewness and Kurtosis.

2.2 An approximation of the distribution of share prices

For our derivation we go back to the starting point, the distribution of share prices. At time t_k the shares have been traded for prices $p_1, \ldots, p_k$. Thus the distribution of prices paid per share is given by some numbers $v_i(t_k)$ specifying the amount of shares for which a price p_i had been paid ($i = 1, \ldots, k$). As before, assume that upon time t_{k+1} some n_{k+1} shares have been traded for a price p_{k+1}. This will alter the distribution of share prices, now given by some numbers $v_i(t_{k+1})$ specifying the amount of share for which a price p_i had been paid ($i = 1, \ldots, k+1$)

Clearly, $v_{k+1}(t_{k+1}) = n_{k+1}$. As in our first derivation (of (3)) we do not know the change from $v_i(t_k)$ to $v_i(t_{k+1})$ for $i \leq k$, since we do not know what the seller originally had paid for his shares. What is known, is that the absolute number of shares floating is constant:

$$\sum_{i=1}^{k} v_i(t_k) = \sum_{i=1}^{n+1} v_i(t_{k+1}) = N. \tag{4}$$

One approximative way to model the evolution of $v_i(t)$ (consistent with the the approximation in the derivation of (3)) is

$$v_i(t_{k+1}) := \frac{N - n_{k+1}}{N} v_i(t_k), \qquad v_{k+1}(t_{k+1}) := n_{k+1} \tag{5}$$

and initially

$$v_1(t_1) := N \tag{6}$$

Note that at time t_{k+1} the mean[3] of the distribution $\{(p_i, v_i(t_{k+1})) \mid 1 \leq i \leq k+1\}$ is

[3] We will drop the tilde and write μ_k, i.e. μ_k is the mean of the approximation of the distribution of prices (in the previous section μ_k was the mean of the real (but unknown) distribution of prices and $\tilde{\mu}_k$ its approximation.

thus given recursively by

$$\begin{aligned}\mu_{k+1} =& \frac{1}{N}\sum_{i=1}^{k+1} v_i(t_{k+1})p_i = \frac{N-n_{k+1}}{N}\frac{1}{N}\sum_{i=1}^{k} v_i(t_k)p_i + \frac{n_{k+1}}{N}p_{k+1} \\ =& \frac{1}{N}\left((N-n_{k+1})\mu_k + n_{k+1}p_{k+1}\right).\end{aligned} \tag{7}$$

Having a model[4] (or approximation) for the evolution of distribution of share prices, we may calculate other statistical measures like standard deviation, Skewness or Kurtosis.

2.3 Standard Deviation, Skewness and Kurtosis

With the notation above, the Standard Deviation σ_k, the Skewness S_k and the Kurtosis K_k of the distribution $\{(p_i, v_i(t_k)) \mid 1 \le i \le k\}$ at time t_k are given by

$$\begin{aligned}\sigma_k &= \left(\frac{1}{N}\sum_{i=1}^{k} v_i(t_k)\cdot(p_i-\mu_k(t_k))^2\right)^{1/2} \\ S_k &= \frac{1}{N}\sum_{i=1}^{k} v_i(t_k)\left(\frac{p_i-\mu_k}{\sigma_k}\right)^3 \\ K_k &= \left(\frac{1}{N}\sum_{i=1}^{k} v_i(t_k)\left(\frac{p_i-\mu_k}{\sigma_k}\right)^4\right) - 3\end{aligned}$$

where μ_k is the mean for which a recursive formula already has been derived above.

To have an algorithm that allows recursive calculation of σ_k and K_k (more precisely: to give these quantities as a function of recursively defined values) let us define

$$\begin{aligned}q_k^{(1)} &= \frac{1}{N}\sum_{i=1}^{k} v_i(t_k)p_i, & q_k^{(2)} &= \frac{1}{N}\sum_{i=1}^{k} v_i(t_k)p_i^2, \\ q_k^{(3)} &= \frac{1}{N}\sum_{i=1}^{k} v_i(t_k)p_i^3, & q_k^{(4)} &= \frac{1}{N}\sum_{i=1}^{k} v_i(t_k)p_i^4,\end{aligned}$$

i.e. recursively (with $v_{k+1}(t_{k+1}) = n_{k+1}$)

$$\begin{aligned}q_0^{(j)} &:= p_0^j \\ q_{k+1}^{(j)} &:= \frac{N-n_{k+1}}{N}q_k^{(j)} + \frac{n_{k+1}}{N}p_{k+1}^j \\ &= \frac{N-n_{k+1}(t_{k+1})}{N}\frac{1}{N}\sum_{i=1}^{k} v_i(t_k)p_i^j + \frac{n_{k+1}}{N}p_{k+1}^j = \frac{1}{N}\sum_{i=1}^{k+1} v_i(t_{k+1})p_i^j\end{aligned}$$

[4]Our model is that at time t_{k+1} the shareholders sell a percentage of $\frac{N-n_{k+1}}{N}$ of their shares independently of what they had originally paid for these shares. This is a crude approximation and other models should be investigated. The choice of the model of course has an influence on the evolution of the distribution and thus on the time series of the mean, standard deviation, etc.

(note that in $q_k^{(j)}$ the j is an index and in p_{k+1}^j the j is an exponent). With these definitions we find that the Standard Deviation, Skewness and Kurtosis of the distribution at any time t_m is given as a function of the $q_m^{(j)}$'s through

$$
\begin{aligned}
\sigma_m &= \left(\frac{1}{N}\sum_{i=1}^{m} v_i \cdot (p_i-\mu_m)^2\right)^{1/2} = \left(q_m^{(2)} - 2\mu_k q_m + \mu_m^2\right)^{1/2} \\
S_m &= \frac{1}{N}\sum_{i=1}^{m} v_i \left(\frac{p_i-\mu_m}{\sigma_m}\right)^3 = \frac{1}{\sigma_m^3}\left(q_m^{(3)} - 3\mu_m q_m^{(2)} + 3\mu_m^2 q_m^{(1)} - \mu_m^3\right) \\
K_m &= \frac{1}{N}\sum_{i=1}^{m} v_i \left(\frac{p_i-\mu_m}{\sigma_m}\right)^4 - 3 \\
&= \frac{1}{\sigma_m^4}\left(q_m^{(4)} - 4\mu_m q_m^{(3)} + 6\mu_m^2 q_m^{(2)} - 4\mu_m^3 q_m^{(1)} + \mu_m^4\right)
\end{aligned}
$$

where μ_m is the mean, i.e. we have $\mu_m = q_m^{(1)}$).

2.4 Considering the time value

So far we have considered the prices paid per share and their distribution given. However it is reasonable to measure the importance (weight) of a price not only by volume but also by time, i.e. to give a lower weight to prices in the past. Instead of introducing an additional time weighting without any further motivation, time weighting can be introduced in a very natural way, namely by replacing all "prices" p by their corresponding time value. This corresponds to the choice of a numeraire. E.g. assuming a constant (continuously compounding) risk free interest rate r, $p_i(t_i)$ should be replaced by $e^{r\cdot(t_i-t_0)} \cdot p_i(t_i)$. This will lead to recursive formulas similar to the ones above, e.g. in analogy to (3) an approximation of the mean of the distribution of the time-value of each share could be given by

$$
\begin{aligned}
\tilde{\mu}_0 &:= p_0 \\
\tilde{\mu}_i &:= \frac{1}{N}\left(N\tilde{\mu}_{i-1}\cdot e^{r\cdot(\Delta t)_i} + n_i(p_i - \tilde{\mu}_{i-1}\cdot e^{r\cdot(\Delta t)_i})\right) \\
&= \frac{1}{N}\left((N-n_i)\tilde{\mu}_{i-1}\cdot e^{r\cdot(\Delta t)_i} + n_i p_i\right).
\end{aligned}
\tag{8}
$$

where in addition to the above r is the risk-free interest rate measuring the time-value of money, $(\Delta t)_i := t_i - t_{i-1}$ is the time between the two transactions (given as fraction of one year) and t_i, t_{i-1} the time of the transaction (p_i, n_i), (p_{i-1}, n_{i-1}), respectively.

3 Random coefficient autoregressive (RCA) models

The time series which appeared so far could be described as *random coefficient autoregressive* (RCA) - a generalization of an AR(1) model (for AR models see e.g. [3, 6]). One of the first in depth studies of RCA models (from a pure mathematical point of view) was performed by Nicholls and Quinn [5]. This short section establishes only the link to between the above models and the RCA models studied in literature. Any further treatment is beyond the scope of this article.

Nicholls and Quinn, e.g, studied models of the form[5]

$$X_i = (A + B_i) \cdot X_{i-1} + \epsilon_i, \tag{9}$$

where X and ϵ_i are p-Variate time-series and A, B_i are $p \times p$-matrices where A is constant and B_i and ϵ_i are independent identically distributed sequences of random variables with mean zero and constant covariance. Furthermore it is assumed that B_i and ϵ_i are independent.

Our models seem to be not of the form studied in [5]. However it is possible to rewrite the time series models which appear here as RCA models which then meet (more or less) the assumptions made in the literature. If we consider a model like (3) and assume that the volume n_i satisfies $n_i = \bar{n} + n_i^0$, such that n_i^0 and $\phi_i = (p_i - (1 + \rho)p_{i-1})$ are independent sequences with mean zero (for some number ρ which may be used to adjust for the expected return in p_i), we may write (replacing $\tilde{\mu}_i$ in (3) by $X_i^{(1)}$)

$$\begin{aligned} X_i^{(1)} &= \frac{N - n_i}{N} \cdot X_{i-1}^{(1)} + \frac{n_i}{N} \cdot p_i \\ &= \left(1 - \frac{\bar{n}}{N} - \frac{n_i^0}{N}\right) \cdot X_{i-1}^{(1)} + \left(\frac{\bar{n}}{N} + \frac{n_i^0}{N}\right) \cdot p_i, \end{aligned}$$

and by defining

$$X_i^{(2)} = (1 + \rho)X_{i-1}^{(2)} + (p_i - (1 + \rho)p_{i-1})$$

we see that the bivariate time series $X := (X^{(1)}, X^{(2)})^T$ is of the form (9) with

$$A = \begin{pmatrix} 1 - \frac{\bar{n}}{N} & \frac{\bar{n}}{N} \\ 0 & 1 + \rho \end{pmatrix} \quad B_i = \begin{pmatrix} -\frac{n_i^0}{N} & +\frac{n_i^0}{N} \\ 0 & 0 \end{pmatrix} \quad \epsilon_i = \begin{pmatrix} 0 \\ p_i - (1 + \rho)p_{i-1} \end{pmatrix}$$

and initial value $X_0 = (p_0, p_0)^T$.

To get the results found for RCA models in literature (like [5]), we have to assume that $n_i - \bar{n}$ and the the increments $p_i - (1+\rho)p_{i-1}$ are independent series of independent identically distributed random numbers with constant covariance and mean zero (for some fixed number n, ρ).

4 Application: Smoothing of time series: (Elastic) Volume Weighted Moving Averages

There is a free parameter in the above model: The total number of shares floating, N. Since the majority of shares is often locked by institutional investors or investors

[5]The models considered in [5] are a bit more general, namely of the form $X_i = \sum_{j=1}^{n} (A_j + B_{i,j}) \cdot X_{i-j} + \epsilon_i$

that do not participate in the marked frequently, it might be well justified to take these shares out of the considerations and consider the distribution of share prices only for the remaining shares.

A first step towards this is to impose a reasonable choice for N, e.g. the total volume traded within the last 15 days[6].

Lowering N will make the recursive update formulas for the distribution (5) or mean approximation (3) more sensitive in the following sense: Assuming that the shares are traded at a constant price p, the mean approximation (3) will converge to this price p as trades continue and the distribution (5) will converge to a one point distribution. Lowering N will increase the convergence speed.

In the following we will take the time series μ_k (which *is* the mean of our model of the distribution of share prices) as a smoothing of the price process p_k and compare this smoothing to standard moving averages of p_k.

To begin, we redefine μ_k as the elastic volume weighted moving average:

4.1 Definition

Let p_0 be the price of a stock at its IPO. Let n_k be the volume and p_k be the price of stocks traded in the n-th transaction. Let N be the total number of shares floating.

Definition 1: The *Elastic Volume Weighted Moving Average* (or *Approximated Average Price Per Share*) is defined by the following recursive formula:

$$\begin{aligned} \mathrm{eVWMA}_0 &:= p_0 \\ \mathrm{eVWMA}_k &:= \mathrm{eVWMA}_{k-1} \cdot (1 - \frac{n_k}{\gamma N}) + \frac{n_k}{\gamma N} \cdot p_k, \end{aligned} \tag{10}$$

where γ is an additional parameter controlling the sensitivity of the average.

Note since for any two values of p_0 the difference of the corresponding sequences of moving average will converge to zero. Thus p_0 may also be approximated by a historical stock price “sufficiently” long ago.

Definition 2: The *Elastic Time and Volume Weighted Moving Average* (or *Approximated Average Time-Price Per Share*) is defined by

$$\begin{aligned} \mathrm{eTVWMA}_0 &:= p_0 \\ \mathrm{eTVWMA}_k &:= \mathrm{eTVWMA}_{k-1} \cdot (1 - \frac{n_k}{\gamma N}) \cdot e^{r \cdot (\Delta t)_k} + \frac{n_k}{\gamma N} p_k, \end{aligned} \tag{11}$$

where in addition to the above r is the risk-free interest rate measuring the time-value of money, $(\Delta t)_k := t_k - t_{k-1}$ is the time between the two transactions (given as fraction of one year) and t_k, t_{k-1} the time of the transaction (p_k, n_k), (p_{k-1}, n_{k-1}), respectively.

It should be noted that the volume weighting makes this averaging especially suited for intraday (tick-by-tick) data.

[6]Such a choice might stem from the assumption that in the average "active market participants" hold shares for 15 days

4.2 Comparison to common moving averages and case study

In this section we will give a brief overview of common moving averages used for smoothing of stock price data (see, e.g., [7]) and give a small case study with real data. However, it is not our intension to perform an in depth analysis of the quality of different moving averages[7]. Neither we want to present the moving average smoothing defined by (10), (11) as something completely new by overemphasizing the differences to other moving averages. Indeed, when the time-series of the volume is almost constant, the difference of (10) to an Exponential Weighted Moving Average (EWMA - see below) is marginal. Thus we believe that the value of this article is not only a refinement of EWMA by introducing volume weighting, but also a justification for EWMA itself as an approximation to eVWMA for stocks with non-volatilty volume.

4.2.1 Common moving averages

The following lists some of the most common moving averages used (see, e.g., [7]):
Standard moving average ($\mathrm{MA}(n)$):
The standard moving average MA is usually defined through

$$\mathrm{MA}(n) = \tfrac{1}{n} \textstyle\sum_{i=1}^{n} p_i.$$

Very common are the 20 day and 30 day moving averages $\mathrm{MA}(20)$ and $\mathrm{MA}(30)$.
Weighted moving average ($\mathrm{WMA}(n)$):
The weighted moving average WMA is usually defined through

$$\mathrm{WMA}(n) = \textstyle\sum_{i=1}^{n} w_i p_i$$

for some weights $w_i > 0$ such that $\sum_{i=1}^{n} w_i = 1$.
Exponential moving average ($\mathrm{EWMA}(w)$):
The exponential weighted moving average[8] EWMA is defined by a recursive formula

$$\mathrm{EWMA}(w)_i = (1 - w) \cdot \mathrm{EWMA}(w)_{i-1} + w \cdot p_i$$

for some weight $0 < w < 1$. It may be *approximated* by a $\mathrm{WMA}(n)$ and weights $w_i = (1 - w)^{n-i} \cdot w$ (if n is sufficiently large).

4.2.2 Example: A small case study

We will now compare the elastic volume weighted moving average to a non-weighted moving average with fixed time period (30 days) and an exponentially weighted moving average with fixed weight (0.95). Although - as should be clear from the introduction - the eVWMA is especially suited for tick-by-tick data and taking daily (closing) price and volume already introduces an inconsistent weighting, we will *not* take the effort to calculate the eVWMA using tick-by-tick data. There are two simple reasons to

[7]It is not even clear what a measure of the quality of the smoothing would be.

[8]EWMAs are beside GARCH models commonly used in volatility modeling, e.g. in J.P. Morgans "RiskMetrics" VAR Methodology, where $\mathrm{EWMA}(0.97)$ is used.

do so: 1. Tick-by-tick data might be not available for the average user and 2. we want to make the comparison fair, by using the same (inconsistent) data (i.e. the closing price) for all three smoothing methods.

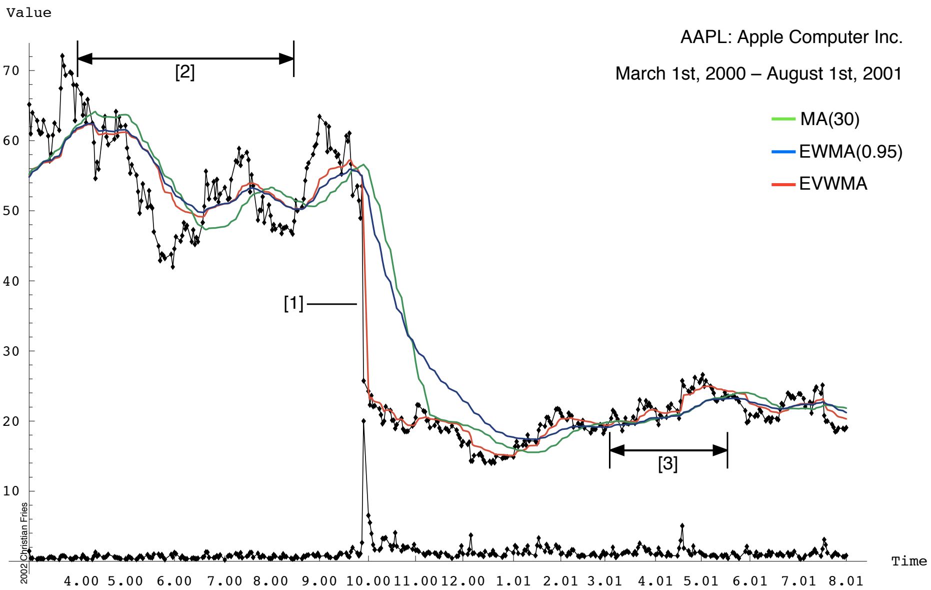

Figure 1: Stock price, eVWMA (red), $\mathrm{MA}(30)$ (green), and EWMA (blue) and Volume[9].

Figure 1 and 2 show the stock price, an unweighted 30-day moving average ($\mathrm{MA}(30)$), an exponentially weighted moving average with weight 0.95 ($\mathrm{EWMA}(0.95)$ and the eVWMA. In Figure 1 at Position (1) we see that the eVWMA changes abruptly and adjusts immediately to the current price due to high volume (volume is plotted at the bottom of the diagram). The high volume (and the sharp price drop) suggest that a strong structural break occurred. Both, MA and EWMA adjust to the new situation with a considerably large delay. The MA adjusts with linear speed, the EWMA with exponential speed. Note that the latter implies that EWMA adjust faster at first, but also slower after a while. By definition, it is clear that EWMA and eVWMA behave similar, when volume is not volatility, as seen at (2). However in (3) we see that MA and EWMA behave almost identical, whereas eVWMA suddenly adjust to a new level due a singular high volume.

In Figure 2, the three averages (MA, EWMA and eVWMA) are shown using different data. In Region (2) and (4) we again see that eVWMA and EWMA behave similarly (which is due to the non-volatile volume). In (1) we see a structural break down similar (but smaller) to the one Figure 1, where MA adjust linearly and EWMA exponentially to the new situation. However this phenomena is not as rare as it might appear and happens also on a small scale, e.g. at (3). (The range (3) is magnified. Note that the averages in the magnification are the same as before and not some adapted "short-range" versions). So even on a small scale eVWMA incorporates small structural breaks (small bursts of bubbles).

[9]Data of Apple Computer Inc., 362 samples (March 1, 2000 to August 1, 2001), price is closing price.

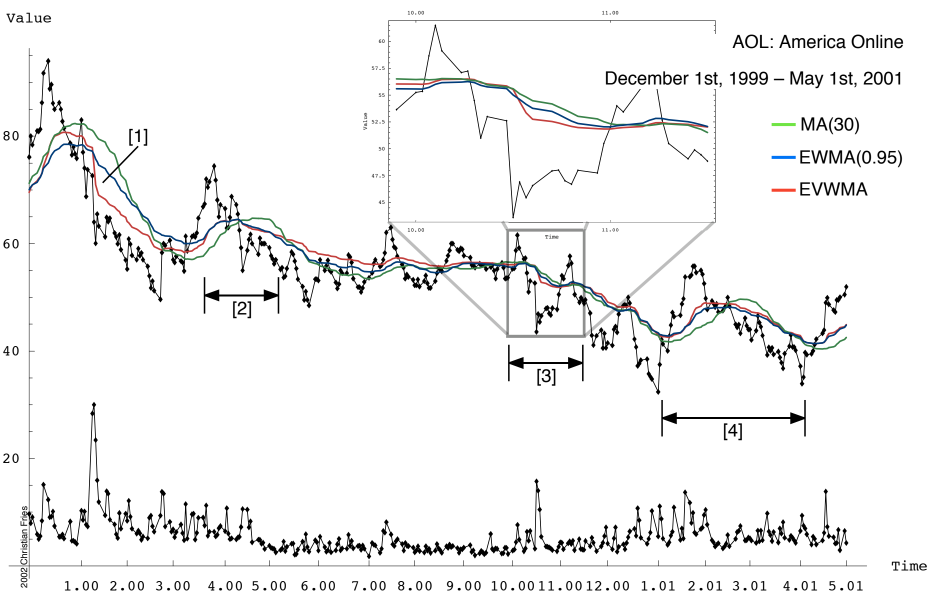

Figure 2: Stock price, eVWMA (red), $\mathrm{MA}(30)$ (green), and EWMA (blue) and Volume[10].

4.3 Defining Bands via Standard Deviation

In the above we used the consideration of the mean of the approximated distribution of share prices to introduce an averaging. In stock price analysis it is common to use some kind of "bands" along the average, i.e. some interval around the average (e.g. so called Bollinger-Bands, cf. [7]). A movement of the stock price out of such a band (i.e. the price deviates to much from the average) is then interpreted in some way.

Since our derivation started from the consideration of a distribution, it is natural to use concepts like standard deviation, confidence level, etc. to construct such "bands" around the mean. E.g. one could recursively calculated the time series which corresponds to the standard deviation in Section 2.3 and add and subtract some fraction of it to the eVWMA.

5 Conclusion & Outlook

In this article we gave a first impression on how the (only theoretically available) *distribution of prices paid per share* can be approximated by a simple model which then gives approximation formulas for statistical measures (e.g. mean, standard deviation), which may suit as replacements for standard moving averages. The resulting smoothing (e.g. the approximated average price paid per share) of the time series of share prices is less synthetic than non-weighted moving averages or volume weighted moving averages with fixed underlying time period. We compared this volume weighted moving average to standard moving averages of stock price time series and derived "higher order" statistical measures.

[10] Data of America Online, 356 samples (December 1, 1999 to May 1, 2001), price is closing price.

We tried to hint towards some of the applications of the volume weighted average in modeling of stock price process, modeling of volatilty, VAR, theory. However an in depth discussion of all of these applications is beyond the scope of this paper.

There are many possible improvements to our approach. Here we want to suggest a few:

1. The modeling of the evolution of the distribution of share prices could be improved, e.g. by introducing a correlation between the price p_i and the change of the corresponding v_i from in (5).

2. The number of shares floating γN, controlling the sensitivity in the "averaging process" could be replaced by a time series itself, e.g. a suitable moving average of volume.

3. The new kinds of elastic volume weighted moving averages could be used to replace (unweighed) averages used in the calculation of so called indicators and thus lead a new zoo of such indicators.

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