Cross Currency LIBOR Models: Backward vs. Forward Algorithm

(A Cross Currency Markov Functional Model)

Christian Fries
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www.christian-fries.de/finmath
Interest Rates
Interest Rates (term structure)

**Zero-Coupon Bond:** Basic Objekt. Interest Rates are seen as derived quantities.

\[ P(T_2) : [0, T_2] \times \Omega \mapsto \mathbb{R} \quad \text{(stochastic process)} \]

\[ P(T_2; t) : \Omega \mapsto \mathbb{R} \quad \text{(random variable):} \]

Value of a guaranteed payment of 1 (unit currency) at time \( T_2 \) as seen in \( t < T_2 \).

Remark: \( P(T_2; T_2) = 1 \)

**Forward-Rate (aka LIBOR):**

\[ \frac{P(T_1)}{P(T_2)} = (1 + L(T_1, T_2) \cdot (T_2 - T_1)). \]

\( L(T_1, T_2; t) : \Omega \mapsto \mathbb{R} \quad \text{(random variable):} \)

Interest rate of 1 invested in \( T_1 \) with payment in \( T_2 \) as seen in \( t \leq T_1 \).

Given \( T_1 < T_2 < \ldots < T_n \) we write \( L_i := L(T_i, T_{i+1}) \) for the stochastic process \( t \mapsto L(T_i, T_{i+1}; t) \).

\[ \Rightarrow \text{term structure: Family of stochastic processes: } L_1, L_2, L_3, \ldots, L_{n-1}. \]
Modeling
Modeling (Itô stochastic process)

Underlying process is modeled as an Itô process:
E.g. lognormal processes for \( L_i \) → LIBOR Market Model:

\[
dL_i(t) = L_i(t)\mu(t)dt + L_i(t)\sigma(t)dW_i,
\]

Time discretization
Given \( 0 = t_0 < t_1 < t_2 < t_3 < \ldots \) use

**Euler Scheme:**

\[
L_i(t_{j+1}) = L_i(t_j) + \mu(t_j) \cdot L_i(t_j) \Delta t_i + \sigma(t_j) \cdot L_i(t_j) \cdot \Delta W_i,
\]

\( \sim \mathcal{N}(0, \sqrt{\Delta t_i}) \)

or

**Euler Scheme of Log-Process:**

\[
L_i(t_{j+1}) = L_i(t_j) \cdot \exp\left((\mu(t_j) - \frac{1}{2} \sigma(t_j)^2) \cdot \Delta t_i + \sigma(t_j) \cdot \Delta W_j\right),
\]

\( \sim \mathcal{N}(0, \sqrt{\Delta t_i}) \)
Pricing
Modeling (Itô stochastic process)

**Example: European option value:** European option value is a known function of the underlying process(es) at some future time $T_{n+1}$:

$$V(T_{n+1}) = \max \left( \left( L_n(T_n) - K \right) \cdot (T_{n+1} - T_n), 0 \right) \quad \text{in } T_{n+1}.$$  

**Universal Pricing Theorem:** Today's value $V(0)$ (the cost of replication) of a derivative is given by the expectation with respect to "some" measure $Q^N$:

$$\frac{V(0)}{N(0)} = E_{Q^N} \left( \frac{V(T_n)}{N(T_n)} \right),$$

where $N$ is some reference product (the so called Numéraire) and $Q^N$ is such that for all financial products the above holds (this characterizes $Q^N$).

For the calculation of the expectation $E_{Q^N}$ it is sufficient to know how the (drift of the) process looks like:

**Definition (Martingal):** A stochastic process $M_t$ is called $\mathbb{P}$-Martingal, if

i) $M_s = E_\mathbb{P}(M_t \mid \mathcal{F}_s) \quad \forall s \leq t$

ii) $E_\mathbb{P}(\mid M_t \mid) < \infty \quad \forall t$

**Martingal property for Itô processes:** Let $X$ denote an Itô stochastic process

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

with $E\left( \left( \int_0^T \sigma_t^2 ds \right)^{1/2} \right) < \infty$. Then

$X$ is a Martingal $\iff$ $X$ has zero drift, i.e. $\mu(t) \equiv 0$. 

\[ 4. \text{Kölner Workshop “Quantitative Finanzmarktforschung”, 13.11.2004 } \]
Example: Calculating the Drift for the LIBOR Market Model

Brace, Gatarek, Musiela; Miltersen, Sandmann, Sondermann; etc.
LIBOR Market Model

Model:
\[ dL_i(t) = L_i(t)\mu_i(t)\,dt + L_i(t)\sigma_i(t)\,dW_i(t) \quad \text{for } i = 0, \ldots, n - 1, \]

Choose some reference product:

E.g. the $T_n$ Bond: $N(t) = P(T_n; t)$ Note: $P(T_n; T_i) = \prod_{k=i}^{n-1} (1 + \delta_k L_k)^{-1}$.

Remember Universal Pricing Theorem / Martingale Property:
\[ \frac{V(0)}{N(0)} = E^{Q_N} \left( \frac{V(T_i)}{N(T_i)} \right), \]

To do:

What does the process $L_i$ look like under the probability measure $Q_N$?
\[ \Leftrightarrow \text{How does } \mu_i \text{ look like?} \]
LIBOR Market Model: Derivation of the Drift under the Pricing Measure (1/3)

Choice of Numeraire: $N(t) = P(T_n; t) (T_n \text{ Bond})$.

Consider the $N$-relative Prices of tradable assets:

$$\frac{P(T_i)}{P(T_n)} = \prod_{k=i}^{n-1} \frac{P(T_k)}{P(T_{k+1})} = \prod_{k=i}^{n-1} (1 + \delta_k L_k) \quad i = 1, \ldots, n-1.$$  

In an arbitrage free model $N$-relative prices are drift free in:

$$\text{Drift} \left[ \frac{P(T_i)}{P(T_n)} \right] = \text{Drift} \left[ \prod_{k=i}^{n-1} (1 + \delta_k L_k) \right] = 0 \quad i = 1, \ldots, n-1 \text{ in } Q^{P(T_n)},$$  

Product rule for Ito processes:

$$d \left( \prod_{k=i}^{n-1} (1 + \delta_k L_k) \right)$$

$$= \prod_{k=i}^{n-1} (1 + \delta_k L_k) \cdot \sum_{j=i}^{n-1} \left( \frac{\delta_j dL_j}{(1 + \delta_j L_j)} + \sum_{l \geq j+1} \frac{\delta_j dL_j}{(1 + \delta_j L_j)} \cdot \frac{\delta_l dL_l}{(1 + \delta_l L_l)} \right).$$
LIBOR Market Model: Derivation of the Drift under the Pricing Measure (2/3)

From
\[
\text{Drift}_{Q^P(T_n)} \left[ \prod_{k=i}^{n-1} (1 + \delta_k L_k) \right] = 0 \quad i = 1, \ldots, n - 1
\]

we find
\[
\sum_{j=i}^{n-1} \text{Drift}_{Q^P(T_n)} \left[ \frac{\delta_j dL_j}{(1 + \delta_j L_j)} + \sum_{l \geq j+1} \frac{\delta_j dL_j}{(1 + \delta_j L_j)} \cdot \frac{\delta_l dL_l}{(1 + \delta_l L_l)} \right] = 0 \quad i = 1, \ldots, n - 1
\]

and thus
\[
\text{Drift}_{Q^P(T_n)} \left[ \frac{\delta_j dL_j}{(1 + \delta_j L_j)} + \sum_{l \geq j+1} \frac{\delta_j dL_j}{(1 + \delta_j L_j)} \cdot \frac{\delta_l dL_l}{(1 + \delta_l L_l)} \right] = 0 \quad j = 1, \ldots, n - 1.
\]
LIBOR Market Model: Derivation of the Drift under the Pricing Measure (3/3)

With

\[ dL_j = L_j \mu_j dt + L_j \sigma_j dW_j, \quad dL_j \cdot dL_l = L_j L_l \sigma_j(t) \sigma_l(t) \rho_{j,l} dt \]

the drift equation

\[
\text{Drift} \quad Q^{P(T_n)} \left[ \frac{\delta_j dL_j}{(1 + \delta_j L_j)} + \sum_{l \geq j+1}^{l \leq n-1} \frac{\delta_j dL_j}{(1 + \delta_j L_j)} \cdot \frac{\delta_l dL_l}{(1 + \delta_l L_l)} \right] = 0 \quad j = 1, \ldots, n-1
\]

becomes

\[
\mu_j \frac{\delta_j L_j}{(1 + \delta_j L_j)} + \sum_{l \geq j+1}^{l \leq n-1} \frac{\delta_j L_j}{(1 + \delta_j L_j)} \cdot \frac{\delta_l L_l}{(1 + \delta_l L_l)} \cdot \sigma_j(t) \sigma_l(t) \rho_{j,l} = 0.
\]

Thus, under the pricing measure \( Q^{P(T_n)} \):

\[
\mu_j = - \sum_{l \geq j+1}^{l \leq n-1} \frac{\delta_l L_l}{(1 + \delta_l L_l)} \cdot \sigma_j(t) \sigma_l(t) \rho_{j,l}
\]

Conclusion:

\[ dL_j = \sum_{j+1 \leq l \leq N-1} \frac{-\delta_l L_l}{(1 + \delta_l L_l)} \cdot \sigma_j(t) \sigma_l(t) \rho_{j,l} dt + L_j \sigma_j(t) dW_j; \quad \text{where} \quad <dW_i, dW_j> = \rho_{i,j} dt \]

Free parameters (→ have to be fitted to market option prices):

\[ \sigma_1(t), \ldots, \sigma_{N-1}(t) \quad \text{(Volatility)}, \quad \rho_{i,j} \quad \text{(Correlation)} \]
Discretization & Implementation (I)

Monte-Carlo / Forward Algorithm
Monte-Carlo / Forward Algorithm

Universal Pricing Theorem:

\[ \frac{V(0)}{N(0)} = E_{Q}^{N}\left(\frac{V(T_{n})}{N(T_{n})}\right) \]

Assume \( V \) consists of finite number of payments

\( X_{i} \) paid in \( T_{i} \), \( X_{i} \) is \( \mathcal{F}_{T_{i}} \)-measurable random variable, \( i = 1, \ldots, n \).

Value

Value of payment:

\[ E_{Q}^{N}\left(\frac{X_{i}}{N(T_{i})}\right) \Rightarrow E_{Q}^{N}\left(\frac{V}{N}\right) = \sum_{i=1}^{n} E_{Q}^{N}\left(\frac{X_{i}}{N(T_{i})}\right) \]

Monte-Carlo Simulation:

Sample space \( \tilde{\Omega} = \{\omega_{1}, \ldots, \omega_{m}\} \), e.g. \( m \approx 10000 \)

\[ E_{Q}^{N}\left(\frac{X_{i}}{N(T_{i})}\right) \approx \frac{1}{m} \sum_{\omega \in \tilde{\Omega}} \frac{X_{i}(\omega)}{N(T_{i}; \omega)} \]

\[ \Rightarrow V(0) = N(0) \cdot E_{Q}^{N}\left(\frac{V}{N}\right) \approx N(0) \cdot \frac{1}{m} \sum_{\omega \in \tilde{\Omega}} \sum_{i=1}^{n} \frac{X_{i}(\omega)}{N(T_{i}; \omega)} \]
Example Product: Bermudan Option
**Bermudan Option**

**Bermudan Option:** Given multiple exercise dates \( T_1 < T_2 < T_3 < \ldots < T_n \) at each time \( T_i \) holder has the choice between

- [exercise] choose the value \( U(T_i) \) of some underlying financial product
- [hold] choose to exercise later, i.e. a Bermudan Option with exercise dates \( \{T_{i+1}, \ldots, T_n\} \).

\[ \rightarrow \text{Option on option ... on option.} \]

**Value of Bermudan Option** according to optimal exercise

\[
V_{\{T_i,\ldots,T_n\}}(T_i) = \max\{U(T_i), V_{\{T_{i+1},\ldots,T_n\}}(T_i)\},
\]

where

\[
V_{\{T_{i+1},\ldots,T_n\}}(T_i) = N(T_i) \cdot \mathbb{E}^{Q_N} \left( \frac{V_{\{T_{i+1},\ldots,T_n\}}(T_{i+1})}{N(T_{i+1})} \mid \mathcal{F}_{T_i} \right)
\]
Bermudan Option

Bermudan Option: Given multiple exercise dates $T_1 < T_2 < T_3 < \ldots < T_n$ at each time $T_i$, holder has the choice between

- choose the value $U(T_i)$ of some underlying financial product
- choose to exercise later, i.e. an Bermudan Option with exercise dates $\{T_{i+1}, \ldots, T_n\}$.

→ Option on option ... on option.

Value of Bermudan Option according to optimal exercise

$$V_{\{T_i, \ldots, T_n\}}(T_i) = \max\{U(T_i), V_{\{T_{i+1}, \ldots, T_n\}}(T_i)\},$$

where

$$V_{\{T_{i+1}, \ldots, T_n\}}(T_i) = N(T_i) \cdot E_Q^N \left( \frac{V_{\{T_{i+1}, \ldots, T_n\}}(T_{i+1})}{N(T_{i+1})} \middle| \mathcal{F}_{T_i} \right)$$

Requires calculation of conditional expectation at some future time
Example Product:
Bermudan Power Reverse Dual Currency
Bermudan Option

**Underlying:**

\[
\max \left( \min \left( \frac{FX(t)}{FX(0)} \cdot A\% - B\% \right), \text{cap} \right), \text{floor} \) - L(t)
\]

where \( A\% \), \( B\% \), cap, floor are constants.

**Product:** A Bermudan Option on the above.

**Requires:**

- **Cross Currency Model**
  - Depends on FX rate
  - Depends on Interest Rates
- **Estimator for Conditional Expectation**
  - Bermudan Option
Cross Currency

LIBOR Market Model
Cross Currency LIBOR Market Model

Model Framework:

Dynamic under $\mathbb{Q}^N$ (with Numéraire (reference product) $N(T_i) := \prod_{j=0}^{i-1}(1 + L_j(T_j) \cdot \delta_j)$:

$$dL_j(t) = L_j(t) \cdot \mu_j(t) \, dt + L_j(t) \cdot \sigma_j(t) \, dW_j \quad j = 1, \ldots, N$$

$$dFX(t) = FX(t) \cdot \mu^{FX}(t) \, dt + FX \cdot \sigma^{FX}(t) \, dW^{FX}$$

$$d\tilde{L}_j(t) = \tilde{L}_j(t) \cdot \tilde{\mu}_j(t) \, dt + \tilde{L}_j(t) \cdot \tilde{\sigma}_j(t) \, d\tilde{W}_j \quad j = 1, \ldots, N$$

where $(\beta(t) := \max\{i \mid T_i \leq t\})$

$$\mu_j(t) = \sum_{\beta(t) \leq l \leq j} \frac{L_l \cdot \delta_l}{1 + L_l \cdot \delta_l} \cdot \sigma_j(t) \sigma_l(t) \rho_{j,l} \int_{T_l}^{T_i} \mu^{FX}(t) \, dt = \log \left( \frac{1 + L_i(T_i) \delta_i}{1 + L_i(T_i) \delta_i} \right)$$

$$\tilde{\mu}_j(t) = \sum_{\beta(t) \leq l \leq j} \frac{\tilde{L}_l \cdot \delta_l}{1 + \tilde{L}_l \cdot \delta_l} \cdot \tilde{\sigma}_j(t) \tilde{\sigma}_l(t) \rho_{j,l} - \tilde{\sigma}_j(t) \sigma^{FX}(t) \rho_{j,FX}(t)$$

Free parameters:

- Rich volatility structure possible: $\sigma_1(t), \ldots, \sigma_{N-1}(t), \bar{\sigma}^F(t), \bar{\sigma}(t), \ldots, \bar{\sigma}_{N-1}(t)$

- Rich correlation structure possible:

  \begin{align*}
  \rho_{i,j}(t) \, dt & = \langle dW_i, dW_j \rangle, \quad \rho_{i,FX}(t) \, dt = \langle dW_i, dW^{FX} \rangle, \quad \rho_{i,FX}(t) \, dt = \langle dW_i, dW^{FX} \rangle, \\
  \rho_{i,j}(t) \, dt & = \langle dW_i, dW_j \rangle, \quad \rho_{i,FX}(t) \, dt = \langle dW_i, dW^{FX} \rangle, \quad \rho_{i,FX}(t) \, dt = \langle dW_i, dW^{FX} \rangle
  \end{align*}

**To do:** Calibrate the model, choose free model parameters such that given/known option prices are reproduced → inverse problem (not always trivial)
Cross Currency LIBOR Market Model

• Characteristics

  • Rich correlation and volatility structure:
    best for correlation sensitive products

  • Markovian only in high dimensions:
    Monte-Carlo Simulation seems natural. Lattice Implementation not trivial.

  • Needs extensions to calibrate smile-surface (ie. to calibrate to more than one interest rate option per maturity)
Discretization & Implementation (2)

Lattice / Backward Algorithm
Lattice / Backward Algorithm

Universal Pricing Theorem:
\[
\frac{V(T_i)}{N(T_i)} = \mathbb{E}^{Q_N} \left( \frac{V(T_{i+1})}{N(T_{i+1})} \mid \mathcal{F}_T \right) \quad \Rightarrow \quad V(T_i) = \mathbb{E}^{Q_N} \left( V(T_{i+1}) \cdot \frac{N(T_i)}{N(T_{i+1})} \mid \mathcal{F}_T \right)
\]

Assume \(V\) (and \(\frac{N(T_i)}{N(T_{i+1})}\)) solely depend on some underlying process \(L\) (e.g. an interest rate), i.e.
\[
V(T_i) = V(T_i, L(T_i))
\]

Expectation conditioned to state variable \(L\)
\[
V(T_i, L^*) = \mathbb{E}^{Q_N} \left( V(T_{i+1}, L(T_{i+1})) \cdot \frac{N(T_i)}{N(T_{i+1})} \mid L(T_i) = L^* \right)
\]

Lattice Method:
Sample state space
\[
\mathbb{E}^{Q_N} \left( V(T_{i+1}, L(T_{i+1})) \cdot \frac{N(T_i)}{N(T_{i+1})} \mid L(T_i) = L^* \right)
\]
\[
\approx \sum_k V(T_{i+1}, L_k) \cdot \frac{N(T_i)}{N(T_{i+1})} \cdot q_{i,k}
\]
where \(q_{i,k}\) denotes the (discrete state) transition probability \(L^* \rightarrow L_k\) from \(T_i\) to \(T_{i+1}\).
Cross Currency Markov Functional Model

Lattice / Backward Algorithm

Paper available on the web:
Fries, Christian P.; Rott, Marius G.:
"Cross Currency and Hybrid Markov Functional Models (2004)"
Cross Currency Markov Functional Model

**Markov Functional Modeling:**
The financial quantities are functions of *some* underlying Markov process.

**Given driving Markov processes:**

\[
\begin{align*}
    dx &= \sigma_x(t) dW_1 \\
    dy &= \mu(t, x, y, z) dt + \sigma_y(t) dW_2 \\
    dz &= \sigma_z(t) dW_3
\end{align*}
\]

**Postulate**

- The *(domestic)* LIBOR \( L(T_k) = L_k(T_k) \) (seen upon its maturity) is a (deterministic) function of \( x(T_k) \): \( L_k(T_k) = L(T_k, x(T_k)) \).

- The FX rate \( FX(T_k) \) is a deterministic function of \( y(T_k) \) only.

- The *(foreign)* LIBOR \( \bar{L}(T_k) = \bar{L}_k(T_k) \) (seen upon its maturity) is a (deterministic) function of \( z(T_k) \).

\[\Rightarrow\] This *forces* a cross-currency LIBOR model onto a computational feasible (3D) lattice (backward algorithm possible).
Markov Functional Model

Markov Functional Modeling:
The financial quantities are functions of some underlying markov process.

1-D (single currency) LIBOR Markov functional model:

Driving Process
\[ dx = \sigma_x(t) dW_1 \]

LIBOR functional
\[ L_k(T_k) = L(T_k, x(T_k)) \]

Where is the pricing measure? What is the Numéraire?

The approach is reversed here: We choose \( N(t) := P(T_n; t) - T_n\)-Bond – as Numéraire.

Then: We postulate that the driving processes are given under the pricing measure \( Q^N \) and define the Numéraire process
\[
\frac{P(T_{i+1}; T_i)}{N(T_i)} := \mathbb{E}^{Q^N} \left( \frac{1}{N(T_{i+1})} | F_{T_i} \right) \Rightarrow N(T_i) := \frac{1}{(1 + L(T_k) \Delta T_i) \cdot \mathbb{E}^{Q^N} \left( \frac{1}{N(T_{i+1})} | F_{T_i} \right)}
\]

→ The numéraire is defined (via backward induction) as a functional of \( x \) too.

The Universal Pricing Theorem holds \textit{per definition}:
\[
\frac{V(T_i)}{N(T_i)} = \mathbb{E}^{Q^N} \left( \frac{V(T_{i+1})}{N(T_{i+1})} | F_{T_i} \right)
\]
Cross Currency Markov Functional Model: Calibration of (domestic) LIBOR Fct'al

Calibration of the (domestic) LIBOR Markov Functional Model

Degrees of Freedom:
- The functional $\xi \mapsto L_i(\xi)$.
- The volatility $\sigma(t)$ of the driving process $x(t)$

Calibration (e.g.) means: The model should reproduce all European Options on $L_i$, i.e. all Product $V$ with

$$V(K; T_{n+1}) = \max \left( (L_n(T_n) - K) \cdot (T_{n+1} - T_n), 0 \right) \quad \text{in } T_{n+1}.$$ 

Calibration to continuous family of European Options on $L_i$ (Caplet smile) can be obtained (almost) explicitly from market prices:

Lemma (Breeden & Litzenberger, etc.): For the probability distribution $\phi_{L_i}$ of $L$ under the measure $Q^N, \ N = P(T_{n+1})$ we have

$$\int_0^K \phi_{L_i}(\kappa) \, d\kappa = N(0) \cdot \frac{\partial}{\partial K} V(K; T_{n+1})$$

$\Rightarrow$ Determination of the functional $\xi \mapsto L_i(\xi)$ can be done through a simple inversion (numerically feasable 1D root finding).
Cross Currency Markov Functional Model: Calibration of FX Functional

Calibration of the FX Functional

Degrees of Freedom:
- The functional \( \eta \mapsto FX(\eta) \).
- The volatility \( \sigma_y(t) \) of the driving process \( y(t) \):
  \[
  dy = \mu(t, x, y, z)dt + \sigma_y(t)dW_2
  \]

Calibration (e.g.) means: The model should reproduce all European FX Options, i.e. all Product \( V \) with

\[
V(K; T_n) = \max \left( (FX(T_n) - K), 0 \right).
\]

1. Drift of the underlying process \( y \)

Example: Choose the functional as \( FX(\eta) = \exp(a \cdot \eta) \rightarrow \) log-normal model for the FX.

Then the drift \( \mu(t, x, y, z) \) has to fulfill

\[
\mu(T_i, \xi, \eta) \Delta T_i = \frac{1}{a} \log \left( \frac{1 + L_i(\xi) \Delta T_i}{1 + \tilde{L}_i(\eta) \Delta T_i} \right) - \frac{a \cdot \sigma_y(T_i)}{2} \Delta T_i
\]

(Note: Same drift adjustment as for the cross currency LIBOR Market Model).
Cross Currency Markov Functional Model: Conclusion

Conclusion

• **Cross Currency MF Model has: Efficient Calibration** (fitting the model to the market)
  - Calibrate to a one parameter family of domestic interest rate option per maturity (e.g. full caplet smile calibration)
  - Calibrate to a one parameter family of foreign interest rate option per maturity (e.g. full caplet smile calibration)
  - Calibrate to “some” (a least one ;-) FX options per maturity

• **Cross Currency MF Model needs:**
  - A fast algorithm for conditional expectation

\[
\mathbb{E}^{Q_N} \left( f(x(T_{i+1}), y(T_{i+1}), z(T_{i+1})) \mid (x(T_i), y(T_i), z(T_i)) = (x^*, y^*, z^*) \right)
\]

Consider: Tree, PDE, Full Numerical Integration, Fourier or Wavelet Methods