7.4.3.1 Example: Time-Discrete Delta-Gamma Hedge under a Black-Scholes Model

We consider a Black-Scholes model with the notation as above. Let C denote an option with maturity T^* and payoff profile $\max(S(T^*) - K^*, 0)$. We aim to replicate an option V with maturity $T < T^*$ and payoff profile $\max(S(T) - K, 0)$. We allow trading at discrete times $0 = t_0 < t_1 < \ldots$.

For the option V to be replicated we have

$$\Delta V(t_k) = \frac{\partial V(t_k)}{\partial t}\,\Delta t_k + \frac{\partial V(t_k)}{\partial S}\,\Delta S(t_k) + \frac{1}{2}\frac{\partial^2 V(t_k)}{\partial S^2}(\Delta S(t_k))^2 + O(|\Delta t_k|^2, |\Delta t_k\,\Delta S(t_k)|, |\Delta S(t_k)|^3).$$

For the replication portfolio Π we have

$$\begin{aligned}\Delta\Pi(t_k) &= \phi_0(t)\,\Delta N(t_k) + \phi_1(t_k)\,\Delta S(t) + \psi_1(t_k)\,\Delta C(t_k)\\ &= \phi_0(t)\,\Delta N(t_k) + \phi_1(t_k)\,\Delta S(t) + \psi_1(t_k)\frac{\partial C(t_k)}{\partial t}\,\Delta t_k\\ &\quad + \psi_1(t_k)\frac{\partial C(t_k)}{\partial S}\,\Delta S(t_k) + \psi_1(t_k)\frac{1}{2}\frac{\partial^2 C(t_k)}{\partial S^2}(\Delta S(t_k))^2\\ &\quad + O(|\Delta t_k|^2, |\Delta t_k\,\Delta S(t_k)|, |\Delta S(t_k)|^3).\end{aligned}$$

For the replication portfolio $\Pi(t) = \phi_0(t)N(t) + \phi_1(t)S(t) + \psi_1(t)C(t)$ we find

$$\begin{aligned}\psi_1(t) &= \frac{\partial^2 V(t)}{\partial S^2(t)}\Big/\frac{\partial^2 C(t)}{\partial S^2(t)} = \frac{\sqrt{T^*}}{\sqrt{T}}\frac{\Phi'(d_+(T,K;t))}{\Phi'(d_+(T^*,K^*;t))}\\ \phi_1(t) &= \frac{\partial V(t)}{\partial S(t)} - \psi_1(t)\frac{\partial C(t)}{\partial S(t)} = \Phi(d_+(T,K;t)) - \psi_1(t)\Phi(d_+(T^*,K^*;t)),\end{aligned}$$

where

$$d_+(T,K;t) := \frac{1}{\sigma\sqrt{T-t}}\left(\log\left(\frac{S(t)}{K}\right) + r(T-t) + \frac{\sigma^2}{2}(T-t)\right).$$

We choose this to trade within the replication portfolio at discrete times t_k. The size of the "cash position" is chosen such that the portfolio remains self-financing, i.e.,

$$\phi_0(t_k) = \frac{1}{N(t_k)}(\phi_0(t_{k-1})N(t_k) + \phi_1(t_{k-1})S(t_k) - \phi_1(t_k)S(t_k) + \psi_1(t_{k-1})C(t_k) - \psi_1(t_k)C(t_k))$$

At time $t_0 = 0$ the portfolio is set up according to the option value known from the evaluation formula

$$\phi_0(t_0) = \frac{1}{N(t_0)}(V(t_0) - \phi_1(t_0)S(t_0) - \psi_1(t_0)C(t_0)).$$