Risk Neutral Valuation

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Version 2.2

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Outline

NOTATION

DIFFERENTIAL EQUATIONS

Notation I

Integral:

$$\int_0^{t_n} f(t)\,\mathrm{d}t \approx \sum_{i=0}^{n-1} f(t_i)\cdot\Delta t_i$$

Discrete Interpretation:

- Integral is approximately the sum of $f(t_i)\cdot\Delta t_i$.
- For piecewise constant function f, constant on $[t_i, t_{i+1})$ we have "‘$=$"’ above.

Notation II

Differential Equation: Define the function $t \mapsto X(t)$ by specifying its change:

$$\mathrm{d}X = f(t)\,\mathrm{d}t \quad :\Leftrightarrow \quad X(T) = X(0) + \int_0^T f(t)\,\mathrm{d}t$$

Discrete Interpretation:

- f is the change of X per unit time (rate of change):
  $\Delta X(t_i) = f(t_i) \cdot \Delta t_i \quad \Leftrightarrow \quad \Delta X(t_i) \big/ \Delta t_i = f(t_i)$.

Notation III

Differential Equation - Special Case: Define the function X by specifying its *relative* change.

$$\mathrm{d}X = f(t) \cdot X(t)\,\mathrm{d}t \quad \Leftrightarrow \quad X(T) = X(0) + \int_0^T f(t) \cdot X(t)\,\mathrm{d}t$$

Discrete Interpretation:

- f is the *relative* change of X per time (percentage rate of change): $\Delta X(t_i) = f(t_i) \cdot X(t_i) \cdot \Delta t_i \quad \Leftrightarrow \quad \frac{\Delta X(t_i)}{X(t_i)} / \Delta t_i = f(t_i)$.

Notation IV

Exercise: Excel Sheet with Discretization of Differential Equation

DifferentialEquation.xls

X(0)	100

t_i	Delta t_i	f(t_i)	X(t_i)	f(t_i) X(t_i) * Delta t_i	X(t_i+1)
0	1	10%	100.00	10.00	110.00
1	1	10%	110.00	11.00	121.00
2	0.5	-10%	121.00	-6.05	114.95
2.5	0.5	-5%	114.95	-2.87	112.08
3	0.5	-20%	112.08	-11.21	100.87
3.5	0.5	5%	100.87	2.52	103.39
4	3	10%	103.39	31.02	134.41
7	3	-5%	134.41	-20.16	114.25
10			114.25		

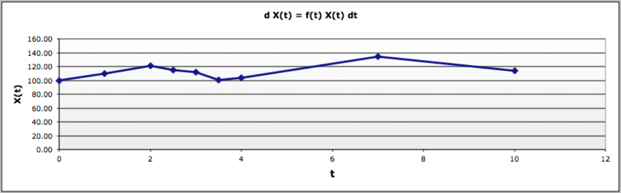

Tabelle1 Tabelle2 Tabelle3

Notation V

State, Pobability, Pobability Measure:

$\Omega = \{ \omega_1, \omega_2, \ldots, \omega_n \}$	(probability space)
$P(\{\omega_i\})$	(probability that we are in state ω_i)
P	(probability measure)

Probability of an State Configuration (Event):

$$P(\{\omega_1, \omega_2, \ldots, \omega_k\}) \; = \; P(\{\omega_1\}) + P(\{\omega_2\}) + \ldots + P(\{\omega_k\})$$

Random Variable:

$X : \Omega \rightarrow \mathbb{R}$ Example: $X(\omega_i)$ payment that depends on the state ω_i.

Expectation:

$$E^P(X) \; := \; \sum_{\omega_i \in \Omega} X(\omega_i) \cdot P(\{\omega_i\})$$

Notation VI

Conditional Expectation:

$$E^P(X|F) := \frac{1}{P(F)} \sum_{\omega_i \in F} X(\omega_i) \cdot P(\{\omega_i\})$$

NOTATION
MONTE-CARLO SIMULATION

Expectation:

$$E^P(X) \ := \ \sum_{\omega_i \in \Omega} X(\omega_i) \cdot P(\{\omega_i\})$$

Monte-Carlo Simulation: Numerical Approximation of Expectation:
Let

$$\tilde{\Omega} := \{ \ \tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_m \ \}$$

denote elements from Ω - *a drawing from* Ω*, i.e. a set of* ***samples***. Some $\tilde{\omega}_i$'s may be the same and we may have more $\tilde{\omega}_i$'s than Ω has elements. Our set of sample paths should have the following property:

A path $\omega \in \Omega$ *occures in* $\tilde{\Omega}$ *approximately* $m \cdot P(\{\omega\})$ *times.*

Then we have

$$E^P(X) \ := \ \sum_{\omega_i \in \Omega} X(\omega_i) \cdot P(\{\omega_i\}) \approx \frac{1}{m} \sum_{\tilde{\omega}_i \in \tilde{\Omega}} X(\tilde{\omega}_i)$$

Outline

Financial Product: A stream of payments depending on events described in a contract.

Modeling:
Value / Payment $\rightarrow$ *Random Variable*

$$X : \Omega \rightarrow \mathbb{R}.$$

- Value of an asset (aka. underlying) at a fixed time t dep. on the states of the world.
- Payment of a financial product at a fixed time t dep. on state of underlying assets.
- Value of the financial product at a fixed time t depending on states of the assets.

Modeling:
Value Process / Payoff Stream $\rightarrow$ *Stochastic Process*
$:=$ *family of random variables over time*

$$S : [0, \infty) \times \Omega \rightarrow \mathbb{R}$$

$S(\omega)$ with $\omega \varepsilon \Omega$ is a map $[0, \infty) \rightarrow \mathbb{R}$: the *path* of S in state ω. Note: All random variables (at all times t) are given over the same space $(\Omega, \mathscr{F})$

$$\omega \in \Omega \quad \leftrightarrow \quad \text{path / history / chain of events}$$

Modeling:

Information is modeled through the *filtration*:

- Family of σ algebras $\mathcal{F}_t$.
- $\mathcal{F}_t \subset \mathcal{F}_s$, $t < s$.
- Elements of $\mathcal{F}_t$ are *the events, which may be known at time* t.

Measurable:

X is $\mathcal{F}_T$ measurable $\quad \leftrightarrow \quad$ X is known in time T

Natural Condition (for stochastic process):

$S(T)$ is $\mathcal{F}_T$ measurable.

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Example: (Time discrete) stochastic process of coin toss at T_1, T_2, T_3.
Modeling (e.g. a bet):

$\Omega = \{(h,h,h),(h,h,t),(h,t,h),\ldots,(t,t,t)\}$ - Prob' space (**h**ead or **t**ail).

$X : \Omega \mapsto \mathbb{R}$ - Bet with a single payoff

$S : \{T_1, T_2, T_3\} \times \Omega \mapsto \mathbb{R}$

$S(T_k)$ paid at time T_k.

} - Bet with payments or evolution of value (stochastic process)

Natural Condition (*): $S(T_k)$ may depend on events known on or before T_k only.

$$\Leftrightarrow \qquad S(T_k, (e_1, \ldots, e_n)) = \text{ const. } \forall e_i \in \{h,t\} \text{ mit } i > k$$

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$$\mathscr{F}_0 = \{0, \Omega\}$$
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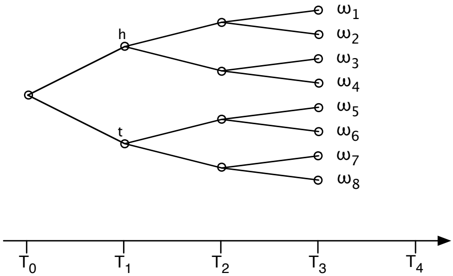

Condition (*) $\Leftrightarrow$ $S(T_k)$ is $\mathscr{F}_k$-measurable $\Leftrightarrow:$ S is adapted to $\{\mathscr{F}_k\}$.

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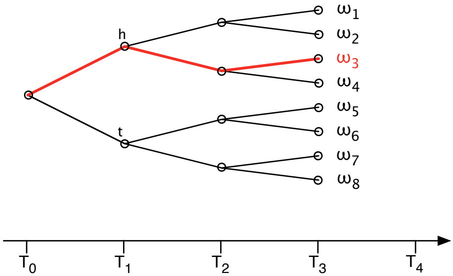

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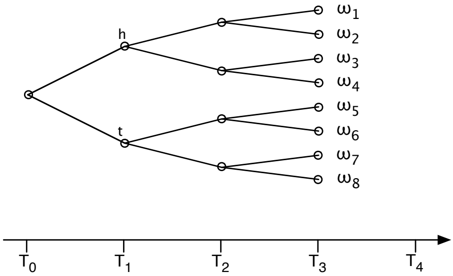

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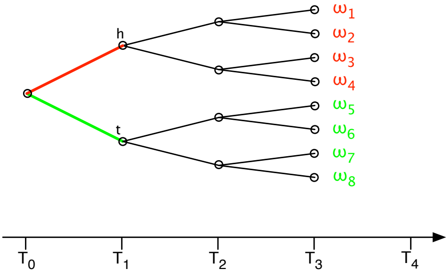

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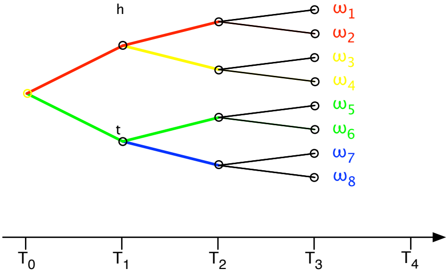

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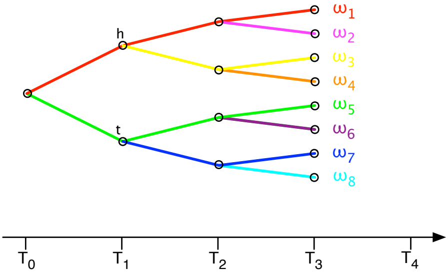

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Modeling:
Prototype of a stochastic process - building block of Itô processes:

Brownian Motion: W

- $W(t)$ defined over $(\Omega, \mathscr{F}, P)$.
- $W(0) = 0$.
- $W(t)$ normal distribution with mean 0 and standard deviation $\sqrt{t}$.
- $W(t_2) - W(t_1)$ normal distribution with mean 0 and standard deviation $\sqrt{t_2 - t_1}$ (i.i.d).
- $W(\cdot, \omega)$ continuous (but nowhere differentiable) function $[0, \infty) \rightarrow \mathbb{R}$ (P-a.s.).

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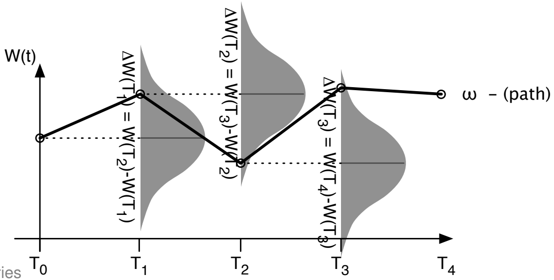
Construction of realizations at discrete times:

$$W(t_k) := \sum_{i=0}^{k-1} \Delta W(t_i) \quad (0 = t_0 < t_1 < \ldots),$$

where

$$W(t_0) := 0 \quad , \qquad \Delta W(t_i) = (W(t_{i+1}) - W(t_i)) \sim \mathcal{N}(0, \sqrt{t_{i+1} - t_i}) \text{ (i.i.d).}$$



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Infinitesimal Notation:

$$W(t) =: \int_0^t \mathrm{d}W(\tau).$$

Stochastic Differential Equation:

$$\mathrm{d}S = \mu(t, S(t))\,\mathrm{d}t \;+\; \sigma(t, S(t))\mathrm{d}W(t)$$

Euler Discretization (e.g.):

$$\underbrace{\Delta \tilde{S}(t_i)}_{\tilde{S}(t_{i+1}) - \tilde{S}(t_i)} = \mu(t_i, \tilde{S}(t_i)) \cdot \underbrace{\Delta t_i}_{t_{i+1} - t_i} + \sigma(t_i, \tilde{S}(t_i)) \cdot \underbrace{\Delta W_i}_{\sim \mathcal{N}(0, \sqrt{\Delta t_i})},$$

i.e.

$$\tilde{S}(t_{i+1}) = \tilde{S}(t_i) + \mu(t_i, \tilde{S}(t_i)) \cdot (t_{i+1} - t_i) \;+\; \sigma(t_i, \tilde{S}(t_i)) \cdot \Delta W_i,$$

with $\tilde{S}(0) = S(0)$.

$\{\tilde{S}(t_i) \mid 0 = t_0 < t_1 < \ldots\}$ is the *Euler discretization* of $\{S(t) \mid t \geq 0\}$.

Example: log normal process for stock value S:

$$\mathrm{d}S(t) = \mu(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t) \quad \Leftrightarrow \quad \frac{\mathrm{d}S(t)}{S(t)} = \mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t)$$

Euler Scheme:

$$\tilde{S}(t_{j+1}) = \tilde{S}(t_j) + \mu(t_j)\cdot\tilde{S}(t_j)\cdot\Delta t_j + \sigma(t_j)\cdot\tilde{S}(t_j)\cdot\underbrace{\Delta W_j}_{\sim\mathcal{N}(0,\sqrt{\Delta t_j})},$$

Log Euler Scheme:

$$\tilde{S}(t_{j+1}) = \tilde{S}(t_j)\cdot\exp\Big(\big(\mu(t_j) - \frac{1}{2}\sigma(t_j)^2\big)\cdot\Delta t_j + \sigma(t_j)\cdot\underbrace{\Delta W_j}_{\sim\mathcal{N}(0,\sqrt{\Delta t_j})}\Big),$$

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Exercise: Excel Sheet with Monte-Carlo Simulation

Monte Carlo Simulation.xls

Model Specification

S(0)	100
r	5%
sigma	10%
delta t	0.01

Model:
Riskless asset: dB = r B dt
Risky asset: dS = r S dt + sigma S dW

	t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12	0.13	
i	B(t)	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.01	1.01	1.01	1.01	
1	S(t,omega_i)	100.00	100.88	102.15	102.97	99.96	99.77	99.00	99.06	99.65	99.61	100.56	99.29	101.46	102.38	10
2	S(t,omega_i)	100.00	100.46	102.59	102.74	100.33	100.24	100.35	100.56	99.71	99.65	101.26	100.91	101.05	100.98	9
3	S(t,omega_i)	100.00	100.89	102.05	102.72	102.86	102.90	103.59	105.15	103.55	103.88	104.60	104.80	105.05	103.27	10
4	S(t,omega_i)	100.00	98.92	98.02	99.19	97.90	97.45	97.48	97.32	97.09	98.72	98.57	99.17	97.51	99.49	9
5	S(t,omega_i)	100.00	103.45	104.52	104.82	104.90	103.31	103.49	103.59	103.15	102.90	102.92	102.02	101.86	102.08	10
6	S(t,omega_i)	100.00	99.79	97.37	98.67	98.16	97.35	96.81	96.97	95.84	95.08	94.56	94.89	95.07	93.90	9
7	S(t,omega_i)	100.00	101.60	102.72	102.13	101.59	102.42	102.62	104.15	102.75	101.54	102.44	102.42	103.17	100.84	10
8	S(t,omega_i)	100.00	100.34	102.48	100.81	100.60	100.21	99.86	101.73	102.07	101.77	100.47	102.17	101.54	101.66	10
9	S(t,omega_i)	100.00	101.18	101.97	100.63	100.95	103.77	105.26	106.90	105.85	106.92	105.26	105.90	104.71	105.23	10
10	S(t,omega_i)	100.00	99.70	101.60	103.03	104.66	105.23	107.21	107.84	107.88	108.14	109.59	111.36	110.96	111.41	11
11	S(t,omega_i)						99.69	99.21	99.78	100.66	100.83	100.01	98.78	99.28	98.31	9
12	S(t,omega_i)						102.84	102.27	101.65	96.39	96.40	98.82	98.45	97.38	96.64	9
13	S(t,omega_i)						98.47	97.93	97.42	102.28	100.33	98.90	97.63	96.98	96.92	9
14	S(t,omega_i)						99.01	99.77	100.83	104.84	104.00	103.81	103.55	103.57	103.59	10
15	S(t,omega_i)						99.80	98.38	97.83	96.35	95.74	95.70	95.85	94.64	95.08	9
16	S(t,omega_i)						97.82	99.74	99.20	99.77	100.21	99.45	100.34	99.74	99.16	10
17	S(t,omega_i)						97.03	97.04	96.25	97.43	97.02	97.68	99.18	99.72	100.00	9
18	S(t,omega_i)						99.54	99.64	99.10	98.70	97.21	97.03	96.97	96.38	96.84	9
19	S(t,omega_i)						99.84	99.69	100.88	102.03	102.35	103.78	105.25	106.15	106.76	10
20	S(t,omega_i)						104.32	104.02	104.46	104.24	104.23	105.31	105.68	105.77	105.72	10
21	S(t,omega_i)						101.28	102.75	102.77	103.22	103.30	103.07	103.53	102.90	102.73	10
22	S(t,omega_i)						99.51	100.88	101.30	103.02	102.63	102.43	103.68	102.66	101.89	10
23	S(t,omega_i)						97.91	99.42	100.45	102.17	102.10	101.32	102.22	103.13	104.86	10
24	S(t,omega_i)						102.16	103.82	104.02	103.82	102.34	104.18	106.91	108.76	108.46	10
25	S(t,omega_i)						104.12	106.07	106.16	106.85	107.22	107.08	107.20	107.90	107.82	10
26	S(t,omega_i)						101.25	101.06	102.71	101.29	100.87	100.18	99.67	100.19	100.82	10
27	S(t,omega_i)						102.19	101.17	101.36	101.61	103.07	103.98	106.63	105.99	105.03	10
28	S(t,omega_i)						97.39	98.64	97.93	98.59	96.92	99.47	97.06	95.32	94.41	9
29	S(t,omega_i)	100.00	100.98	100.57	100.61	100.62	100.61	101.10	101.23	102.20	101.54	102.17	104.46	105.05	104.41	10
30	S(t,omega_i)	100.00	101.08	99.81	99.98	99.28	99.86	100.58	100.44	100.94	100.72	100.46	101.40	101.95	101.81	10
31	S(t,omega_i)	100.00	99.28	98.40	96.26	96.49	97.56	98.92	99.57	98.71	99.21	98.50	98.15	97.16	98.03	9
32	S(t,omega_i)	100.00	99.12	100.86	100.24	101.48	101.14	101.93	102.59	104.34	103.25	103.28	103.64	103.44	102.16	10
33	S(t,omega_i)	100.00	99.08	98.54	100.53	99.86	99.90	100.85	101.89	103.84	103.42	102.88	103.95	102.68	103.56	10
34	S(t,omega_i)	100.00	99.16	98.21	99.38	100.49	99.27	99.35	99.78	101.21	102.95	104.88	103.09	105.09	105.71	10
35	S(t,omega_i)	100.00	101.21	103.68	101.86	101.07	100.89	102.91	103.99	103.40	101.69	101.94	102.05	103.85	102.58	10
36	S(t,omega_i)	100.00	99.47	99.47	99.36	99.77	100.11	101.02	100.00	98.00	98.00	98.14	98.55	98.01	97.57	9

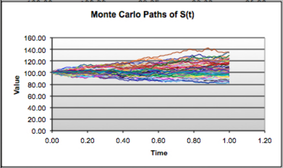

Tabelle1 Tabelle2 Tabelle3
Normalansicht Bereit Berechnen Summe= 100

Outline

Product A: *Weather Derivative*
At time $T_2 > 0$ measure amount of rain $R(T_2)$ (in mm) fallen at a predetermined place for a predetermined period. Pay the Euro amount

$$A(T_2) := (R(T_2) - X) \cdot \frac{€}{mm}.$$

What determines the value $A(T_1)$ of this contract at time $T_1 < T_2$?
Note: $R(T_2)$ is stochastic
⇒ Value depends on assessment of the probability of rain, risk, ...

Product B: *Equity Derivative*
At time T_2 measure the quoted value of IBM $S(T_2)$ (in €). Pay the Euro amount

$$B(T_2) := (S(T_2) - X)$$

What determined the value $B(T_1)$ of this contract at time $T_1 < T_2$?
⇒ Product looks most similar to the first one.

Replication:

The payoff of B can be replicated through products traded in T_1:
Let $P(T_2; T_1)$ denote the value of a credit which has to be payed back in T_2 by an amount of 1 ($\rightarrow$ *Zero Bond*). In T_1:

Buy stock $S(T_1)$ and X times a credit $P(T_2; T_1)$
(stock long, bond short).

Then

$$\begin{aligned} &\text{Value in } T_2: && S(T_2) - X && = B(T_2) \\ &\text{Value in } T_1: && S(T_1) - X \cdot P(T_2; T_1) && \stackrel{!}{=} B(T_1) \end{aligned}$$

$\Rightarrow$ Value of replication portfolio is a “fair value” for the derivative.

Discrete Time (T_1, T_2), Two Assets (S, B), Two States (ω_1, ω_2)

Given: In T_2 we have two disjoint events (states) ω_1 and ω_2, the payoff of a derivative product $V(T_2, \omega_i)$ and two traded assets S und B.

Wanted: A portfolio $\alpha S + \beta B$, which replicates the value $V(T_2)$ in T_2 *for any state*.
$\Rightarrow$ Its value in T_1 determines the (fair) value $V(T_1)$ of V
$\rightarrow$ *cost of replication*.

Solution:

$$\alpha \cdot S(T_2;\omega_1) + \beta \cdot B(T_2;\omega_1) \overset{!}{=} V(T_2;\omega_1)$$

$$\alpha \cdot S(T_1) + \beta \cdot B(T_1)$$

$$\alpha \cdot S(T_2;\omega_2) + \beta \cdot B(T_2;\omega_2) \overset{!}{=} V(T_2;\omega_2)$$

Risk Neutral Valuation

Linear equation, two equations, two unknowns.

$$\alpha \cdot S(T_1) + \beta \cdot B(T_1) \;\longrightarrow\; \alpha \cdot S(T_2;\omega_1) + \beta \cdot B(T_2;\omega_1) \stackrel{!}{=} V(T_2;\omega_1)$$

$$\alpha \cdot S(T_1) + \beta \cdot B(T_1) \;\longrightarrow\; \alpha \cdot S(T_2;\omega_2) + \beta \cdot B(T_2;\omega_2) \stackrel{!}{=} V(T_2;\omega_2)$$

Solvable

$$\Leftrightarrow \quad S(T_2;\omega_1) \cdot B(T_2;\omega_2) - S(T_2;\omega_2) \cdot B(T_2;\omega_1) \neq 0$$

$$\stackrel{B \neq 0}{\Leftrightarrow} \quad \frac{S(T_2;\omega_1)}{B(T_2;\omega_1)} \neq \frac{S(T_2;\omega_2)}{B(T_2;\omega_2)}$$

Overhead: For every derivative product V, the replication portfolio $\alpha(t)$, $\beta(t)$ has to be calculated (though every time step t).

Risk Neutral Valuation

Same example again - but consider everything divided by B

$$\alpha \cdot \frac{S(T_1)}{B(T_1)} + \beta \cdot 1$$

p:

$$\alpha \cdot \frac{S(T_2;\omega_1)}{B(T_2;\omega_1)} + \beta \cdot 1 \stackrel{!}{=} \frac{V(T_2;\omega_1)}{B(T_2;\omega_1)}$$

$1-p$:

$$\alpha \cdot \frac{S(T_2;\omega_2)}{B(T_2;\omega_2)} + \beta \cdot 1 \stackrel{!}{=} \frac{V(T_2;\omega_2)}{B(T_2;\omega_2)}$$

Risk Neutral Valuation

Same example again - but consider everything divided by B

$$\alpha \cdot \frac{S(T_1)}{B(T_1)} + \beta \cdot 1$$

$p \longrightarrow$

$$\alpha \cdot \frac{S(T_2;\omega_1)}{B(T_2;\omega_1)} + \beta \cdot 1 \stackrel{!}{=} \frac{V(T_2;\omega_1)}{B(T_2;\omega_1)}$$

$1-p \longrightarrow$

$$\alpha \cdot \frac{S(T_2;\omega_2)}{B(T_2;\omega_2)} + \beta \cdot 1 \stackrel{!}{=} \frac{V(T_2;\omega_2)}{B(T_2;\omega_2)}$$

In general

$$\frac{V(T_1)}{B(T_1)} \neq \mathrm{E}^{\mathbb{P}}\left(\frac{V(T_2)}{B(T_2)}\right) \qquad \text{and thus} \qquad \frac{S(T_1)}{B(T_1)} \neq \mathrm{E}^{\mathbb{P}}\left(\frac{S(T_2)}{B(T_2)}\right).$$

Risk Neutral Valuation

Same example again - but consider everything divided by B

$$\alpha \cdot \frac{S(T_1)}{B(T_1)} + \beta \cdot 1$$

q →

$$\alpha \cdot \frac{S(T_2;\omega_1)}{B(T_2;\omega_1)} + \beta \cdot 1 \overset{!}{=} \frac{V(T_2;\omega_1)}{B(T_2;\omega_1)}$$

$1-q$ →

$$\alpha \cdot \frac{S(T_2;\omega_2)}{B(T_2;\omega_2)} + \beta \cdot 1 \overset{!}{=} \frac{V(T_2;\omega_2)}{B(T_2;\omega_2)}$$

Same example again - but consider everything divided by B

$$\alpha \cdot \frac{S(T_1)}{B(T_1)} + \beta \cdot 1$$

with probability q:

$$\alpha \cdot \frac{S(T_2;\omega_1)}{B(T_2;\omega_1)} + \beta \cdot 1 \stackrel{!}{=} \frac{V(T_2;\omega_1)}{B(T_2;\omega_1)}$$

with probability $1-q$:

$$\alpha \cdot \frac{S(T_2;\omega_2)}{B(T_2;\omega_2)} + \beta \cdot 1 \stackrel{!}{=} \frac{V(T_2;\omega_2)}{B(T_2;\omega_2)}$$

However, if $\mathrm{sign}\left(\frac{S(T_2;\omega_2)}{B(T_2;\omega_2)} - \frac{S(T_1)}{B(T_1)}\right) \neq \mathrm{sign}\left(\frac{S(T_2;\omega_1)}{B(T_2;\omega_1)} - \frac{S(T_1)}{B(T_1)}\right)$ there exists a prob.-measure $\mathbb{Q}$ s.th.

$$\frac{S(T_1)}{B(T_1)} = \mathrm{E}^{\mathbb{Q}}\left(\frac{S(T_2)}{B(T_2)}\right) \qquad \text{and thus} \qquad \frac{V(T_1)}{B(T_1)} = \mathrm{E}^{\mathbb{Q}}\left(\frac{V(T_2)}{B(T_2)}\right).$$

Equivalent Martingale Measure

The measure $\mathbb{Q}$ is the so called *equivalent martingale measure*. It does not depend on V (!) - this does not hold if one considers

$$V(T_1) \stackrel{!}{=} \mathrm{E}(V(T_2)) \text{ in place of } \frac{V(T_1)}{B(T_1)} \stackrel{!}{=} \mathrm{E}\left(\frac{V(T_2)}{B(T_2)}\right).$$

Numéraire

The measure $\mathbb{Q}$ depends on the reference quantity (here B), the so called *numéraire*.

Here:

$$q \cdot \frac{S(T_2;\omega_1)}{B(T_2;\omega_1)} + (1-q) \cdot \frac{S(T_2;\omega_2)}{B(T_2;\omega_2)} \stackrel{!}{=} \frac{S(T_1)}{B(T_1)} \quad \Rightarrow \quad q = \frac{\frac{S(T_2;\omega_1)}{B(T_2;\omega_1)} - \frac{S(T_1)}{B(T_1)}}{\frac{S(T_2;\omega_1)}{B(T_2;\omega_1)} - \frac{S(T_2;\omega_2)}{B(T_2;\omega_2)}}$$

Universal Pricing Theorem:

$$\frac{V(0)}{N(0)} = \mathrm{E}^{\mathbb{Q}^N}\left(\frac{V(T_n)}{N(T_n)} \mid \mathscr{F}_0\right)$$

Assume V consists of finite number of payments

$$X_i \text{ paid in } T_i, \quad \Rightarrow \quad N\text{-relative payment value: } \mathrm{E}^{\mathbb{Q}^N}\left(\frac{X_i}{N(T_i)} \middle| \mathscr{F}_0\right).$$

Value

$$\Rightarrow \quad \mathrm{E}^{\mathbb{Q}^N}\left(\frac{V}{N} \mid \mathscr{F}_0\right) = \sum_{i=1}^{n} \mathrm{E}^{\mathbb{Q}^N}\left(\frac{X_i}{N(T_i)} \mid \mathscr{F}_0\right)$$

Monte-Carlo Simulation:

Sample space $\tilde{\Omega} = \{\omega_1, \ldots, \omega_m\}$, e.g. $m \approx 10000$

- $\frac{V(0)}{N(0)} = \mathrm{E}^{\mathbb{Q}^N}\left(\frac{V}{N} \mid \mathscr{F}_0\right) = \sum_{i=1}^{n} \mathrm{E}^{\mathbb{Q}^N}\left(\frac{X_i}{N(T_i)} \mid \mathscr{F}_0\right)$
- $\mathrm{E}^{\mathbb{Q}^N}\left(\frac{X_i}{N(T_i)} \middle| \mathscr{F}_0\right) \approx \frac{1}{m} \sum_{\omega \in \tilde{\Omega}} \frac{X_i(\omega)}{N(T_i;\omega)}$

$$\Rightarrow \qquad V(0) = N(0) \cdot \mathrm{E}^{\mathbb{Q}^N}\left(\frac{V}{N} \mid \mathscr{F}_0\right) \approx \frac{1}{m} \sum_{\omega \in \tilde{\Omega}} \sum_{i=1}^{n} X_i(\omega) \cdot \underbrace{\frac{N(0)}{N(T_i;\omega)}}_{\text{state price deflator / discount factor}}$$

Monte-Carlo Simulation

Advantages

- Pricing of path dependent options in a natural way.
- High dimensions possible (but slower convergence in high dimensions).
- Straight forward to implement.

Challenges

- Sensitivities (partial derivatives of option price w.r.t. model parameters) tend to be unstable (harder to get them stable).
- Pricing of Bermudan option needs additional effort.
- Calibration.

Monte-Carlo Simulation

Alternatives

- Analytic pricing formulas - only for simple models and simple payoffs.
- Numerical integration - only for simple models and European style options.
- Lattice models (PDE, Tree, Markov Functional) - only for low dimensional models

Risk Neutral Valuation

Change of Measure / Drift

Change of Drift Trick I

Risk Neutral Pricing: Find a measure $\mathbb{Q}$ such that for all underlying assets S we have

$$\mathrm{E}^{\mathbb{Q}}\left(\frac{S(T_{i+1})}{N(T_{i+1})} \mid \mathcal{F}_{T_i}\right) = \frac{S(T_i)}{N(T_i)},$$

then for a derivative the cost of replication V satisfies

$$\mathrm{E}^{\mathbb{Q}}\left(\frac{V(T_{i+1})}{N(T_{i+1})} \mid \mathcal{F}_{T_i}\right) = \frac{V(T_i)}{N(T_i)}.$$

Martingale Property: (in place of $\frac{S}{N}$ or $\frac{V}{N}$ we simple write X)

$$\mathrm{E}^{\mathbb{Q}}\big(X(T_{i+1}) \mid \mathcal{F}_{T_i}\big) = X(T_i) \quad \Leftrightarrow \quad \mathrm{E}^{\mathbb{Q}}\big(\underbrace{X(T_{i+1}) - X(T_i)}_{=:\Delta X(T_i)} \mid \mathcal{F}_{T_i}\big) = 0$$

How to calculate the measure $\mathbb{Q}$?

Change of Drift Trick II

Never need to (explicitly) calculate the measure $\mathbb{Q}$:
Conditional Expectation:

$$\mathrm{E}^{\mathbb{Q}}(\Delta X(T_i) \mid \mathscr{F}_{T_i}) = \sum_{\omega \in \Omega} \Delta X(T_i, \omega) \cdot \mathbb{Q}(\{\omega\})$$

interpretation 1: the measure has changed from $\mathbb{P}$ to $\mathbb{Q}$:

$$= \sum_{\omega \in \Omega} \Delta X(T_i, \omega) \cdot \underbrace{\frac{\mathbb{Q}(\{\omega\})}{\mathbb{P}(\{\omega\})} \mathbb{P}(\{\omega\})}_{\text{measure changed}}$$

interpretation 2: the values have changed:

$$= \sum_{\omega \in \Omega} \underbrace{\Delta X(T_i, \omega) \frac{\mathbb{Q}(\{\omega\})}{\mathbb{P}(\{\omega\})}}_{\text{value changed}} \cdot \mathbb{P}(\{\omega\})$$

Change of Drift Trick III

Question: How does the process look like, if it has to satisfy the martingale property?

Answer: The drift is zero.

$$\left.\begin{aligned} \mathrm{d}X(t) &= \mu \mathrm{d}t + \sigma \mathrm{d}W(t) \\ &\text{and} \\ \mathrm{E}^{\mathbb{Q}}(X(T_{i+1}) - X(T_i) \mid \mathscr{F}_{T_i}) &= 0 \end{aligned}\right\} \quad \Rightarrow \quad \mu = 0$$

Note: $X(T_{i+1}) - X(T_i) = \Delta X = \int_{T_i}^{T_{i+1}} \mathrm{d}X(t)$.

Change of Drift Trick IV

Conclusion:

Instead of calculating the pricing measure (martingale measure), simply write down the process and calculate the correct drift.

$$\text{change of measure} \;\Leftrightarrow\; \text{change of drift}$$

A Monte-Carlo simulation of that process corresponds to a simulation under the pricing measure.

Change of Drift Trick V

Itô Lemma: Main Tool to Calculate the Drift of Processes
Let X denote an Itô prozess with

$$\mathrm{d}X(t) = \mu \, \mathrm{d}t + \sigma \, \mathrm{d}W.$$

Let $g(t,x)$ denote some function $g \in C^2([0,\infty] \times \mathbb{R})$. The we have that

$$Y(t) := g(t, X(t))$$

is an Itô process with

$$\mathrm{d}Y_t = \frac{\partial g}{\partial t}(t, X_t) \, \mathrm{d}t + \frac{\partial g}{\partial x}(t, X_t) \, \mathrm{d}X_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (\mathrm{d}X_t)^2,$$

where $(\mathrm{d}X_t)^2 = (\mathrm{d}X_t) \cdot (\mathrm{d}X_t)$ is given by formal expansion with

$$\mathrm{d}t \cdot \mathrm{d}t = 0, \qquad \mathrm{d}t \cdot \mathrm{d}W = 0,$$
$$\mathrm{d}W \cdot \mathrm{d}t = 0, \qquad \mathrm{d}W \cdot \mathrm{d}W = \mathrm{d}t,$$

Change of Drift Trick VI

i.e.

$$(\mathrm{d}X_t)^2 = (\mathrm{d}X_t) \cdot (\mathrm{d}X_t) = (\mu \ \mathrm{d}t + \sigma \ \mathrm{d}W) \cdot (\mu \ \mathrm{d}t + \sigma \ \mathrm{d}W)$$
$$= \mu^2 \cdot \mathrm{d}t \cdot \mathrm{d}t + \mu \cdot \sigma \cdot \mathrm{d}t \cdot \mathrm{d}W + \mu \cdot \sigma \cdot \mathrm{d}W \cdot \mathrm{d}t + \sigma^2 \cdot \mathrm{d}W \cdot \mathrm{d}W$$
$$= \sigma^2 \ \mathrm{d}t$$

Change of Drift Trick VII

Itô Lemma: Example:

$$\mathrm{d}X(t) = \mu X(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}W.$$

Consider logarithm of X

$$Y(t) := \log(X(t))$$

then

$$\mathrm{d}Y(t) = (\mu - \frac{1}{2}\sigma^2)\,\mathrm{d}t + \sigma\,\mathrm{d}W.$$

Risk Neutral Valuation
Example: Black Scholes Model

Example: Black-Scholes Model

Black-Scholes Model:

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}W \quad \Leftrightarrow \quad \frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}W$$

Interpretation: Consider discrete times $t_0, t_1, t_2, t_3, \ldots$. Then:

$$\frac{\Delta S(t_i)}{S} = \mu \cdot \Delta t_i + \sigma \cdot \Delta W(t_i)$$

where

$$\Delta t_i = t_{i+1} - t_i$$

A time period.

$$\frac{\Delta S(t_i)}{S(t_i)} = \frac{S(t_{i+1}) - S(t_i)}{S(t_i)}$$

Relative change of the stock over the period $\Delta t_i = t_{i+1} - t_i$.

$$\mu \cdot \Delta t_i = \mu \cdot (t_{i+1} - t_i)$$

Risk less part: A deterministic drift. $\mu =$ annualized risk less rate of return.

$$\sigma \cdot \Delta W(t_i) = \sigma \cdot \underbrace{(W(t_{i+1}) - W(t_i))}_{\text{Normal distributed with variance } \Delta t_i}$$

Risky part: A normal distributed random variable. $\sigma =$ annualized standard deviation of the return.

Example: Black-Scholes Model

Black-Scholes Model:

$$\mathrm{d}S = \mu S \mathrm{d}t + \sigma S \mathrm{d}W \quad \Leftrightarrow \quad \frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}W$$

Interpretation: Consider discrete times $t_0, t_1, t_2, t_3, \ldots$. Then:

$$\frac{\Delta S(t_i)}{S} = \mu \cdot \Delta t_i + \sigma \cdot \Delta W(t_i)$$

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A time period.

$$\frac{\Delta S(t_i)}{S(t_i)} = \frac{S(t_{i+1}) - S(t_i)}{S(t_i)}$$

Relative change of the stock over the period $\Delta t_i = t_{i+1} - t_i$.

$$\mu \cdot \Delta t_i = \mu \cdot (t_{i+1} - t_i)$$

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$$\sigma \cdot \Delta W(t_i) = \sigma \cdot \underbrace{(W(t_{i+1}) - W(t_i))}_{\text{Normal distributed with variance } \Delta t_i}$$

Risky part: A normal distributed random variable. $\sigma =$ annualized standard deviation of the return.

Implementation: (Monte-Carlo) Simulation: Consider time-discrete version of Black-Scholes Model

$$\frac{\Delta S(t_i)}{S} = \mu \cdot \Delta t_i + \sigma \cdot \Delta W(t_i)$$

i.e.

$$S(t_{i+1}) \ = \ S(t_i) + \mu \cdot S(t_i) \cdot \Delta t_i + \sigma \cdot S(t_i) \cdot \Delta W(t_i).$$

For a single path (scenario) we have

$$S(t_{i+1}, \omega) \ = \ S(t_i, \omega) + \mu \cdot S(t_i, \omega) \cdot \Delta t_i + \sigma \cdot S(t_i, \omega) \cdot \Delta W(t_i, \omega) \quad \text{(Euler Sch}$$

Remark: Much more accurate numerical simulation is given by

$$S(t_{i+1}, \omega) \ = \ S(t_i, \omega) \cdot \exp\left((\mu - \frac{1}{2}\sigma^2) \cdot \Delta t_i + \sigma \cdot \Delta W(t_i, \omega)\right) \quad \text{(Log-Euler}$$

Implementation: (Binomial) Tree: Consider time-discrete version of Black-Scholes Model

$$S(t_{i+1}) \; = \; S(t_i) + \mu \cdot S(t_i) \cdot \Delta t_i + \sigma \cdot S(t_i) \cdot \Delta W(t_i).$$

Approximate the normal distributed $\Delta W(t_i)$ by a binomial distributed random variable $\Delta B(t_i)$

$$S(t_{i+1}) \; \approx \; S(t_i) + \mu \cdot S(t_i) \cdot \Delta t_i + \sigma \cdot S(t_i) \cdot \Delta B(t_i).$$

where

$$\Delta B(t_i) = \begin{cases} +\sqrt{\Delta t_i} & \text{with probability } \frac{1}{2} \\ -\sqrt{\Delta t_i} & \text{with probability } \frac{1}{2} \end{cases}$$

Justification: Repetitive binomial experiments converge to a normal distribution.

$$\sum_{i=0}^{n} \Delta B(t_i) \; \approx \; \sum_{i=0}^{n} \Delta W(t_i) \quad \text{for } n \text{ large.}$$

Back-Scholes Model:

Evolution of Money Market Account: $\mathrm{d}B = r \cdot B\,\mathrm{d}t$

Evolution of Stock . . . . : $\mathrm{d}S = \mu \cdot S\mathrm{d}t + \sigma \cdot S\mathrm{d}W(t)$

Choose Numéraire and consider Martingale Measure:

B chosen as Numéraire $\Rightarrow$ $\frac{S}{B}$ is $\mathbb{Q}$-martingale

Change of Drift Trick:

$$\mathrm{d}\frac{S}{B} = (\mu - r)\frac{S}{B}\,\mathrm{d}t + \sigma\frac{S}{B}\mathrm{d}W(t)$$

drift has to be zero

$$\mu = r$$

RISK NEUTRAL VALUATION

EXAMPLE: BLACK-SCHOLES MODEL
MONTE-CARLO OPTION PRICER

Example: European option value: European option value is a known function of the underlying process(es) at some future time T_n:

$$V(T_n) = \max(S(T_n) - K, 0).$$

Choose a Model for Underlyings: e.g. Black-Scholes

Evolution of Money Market Account: $\mathrm{d}B = r \cdot B \, \mathrm{d}t$

Evolution of Stock .... : $\mathrm{d}S = \mu \cdot S \mathrm{d}t + \sigma \cdot S \mathrm{d}W(t)$

Choose Numéraire and consider Martingale Measure:

$$B \text{ Numéraire} \Rightarrow \frac{S}{B} \ \mathbb{Q}\text{-martingale} \Rightarrow \mu = r$$

Monte-Carlo Simulation (of the $\mathbb{Q}$ dynamics):

$$B(t_{i+1}, \omega) = B(t_0) \cdot \exp(r \cdot t_{i+1})$$
$$S(t_{i+1}, \omega) = S(t_i) + r \cdot S(t_i) + \sigma \cdot S(t_i) \cdot \Delta W(t_i, \omega)$$

for $\omega = \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \ldots, \omega_N$ (some 1000's samples).

Example: European option value: European option value is a known function of the underlying process(es) at some future time T_n:

$$V(T_n) = \max(S(T_n) - K, 0).$$

Choose a Model for Underlyings: e.g. Black-Scholes

Evolution of Money Market Account: $\mathrm{d}B = r \cdot B \, \mathrm{d}t$

Evolution of Stock .... : $\mathrm{d}S = \mu \cdot S \mathrm{d}t + \sigma \cdot S \mathrm{d}W(t)$

Choose Numéraire and consider Martingale Measure:

$$B \text{ Numéraire } \Rightarrow \frac{S}{B} \ \mathbb{Q}\text{-martingale} \Rightarrow \mu = r$$

Monte-Carlo Simulation (of the $\mathbb{Q}$ dynamics):

$$B(t_{i+1}, \omega) = B(t_0) \cdot \exp(r \cdot t_{i+1})$$

$$S(t_{i+1}, \omega) = S(t_i) + r \cdot S(t_i) + \sigma \cdot S(t_i) \cdot \Delta W(t_i, \omega)$$

for $\omega = \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \ldots, \omega_N$ (some 1000's samples).

Example: European option value: European option value is a known function of the underlying process(es) at some future time T_n:

$$V(T_n) = \max(S(T_n) - K, 0).$$

Choose a Model for Underlyings: e.g. Black-Scholes

Evolution of Money Market Account: $\mathrm{d}B = r \cdot B \, \mathrm{d}t$

Evolution of Stock . . . . : $\mathrm{d}S = \mu \cdot S \mathrm{d}t + \sigma \cdot S \mathrm{d}W(t)$

Choose Numéraire and consider Martingale Measure:

$$B \text{ Numéraire} \Rightarrow \frac{S}{B} \ \mathbb{Q}\text{-martingale} \Rightarrow \mu = r$$

Monte-Carlo Simulation (of the $\mathbb{Q}$ dynamics):

$$B(t_{i+1}, \omega) = B(t_0) \cdot \exp(r \cdot t_{i+1})$$
$$S(t_{i+1}, \omega) = S(t_i) + r \cdot S(t_i) + \sigma \cdot S(t_i) \cdot \Delta W(t_i, \omega)$$

for $\omega = \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \ldots, \omega_N$ (some 1000's samples).

Example: European option value: European option value is a known function of the underlying process(es) at some future time T_n:

$$V(T_n) = \max(S(T_n) - K, 0).$$

Choose a Model for Underlyings: e.g. Black-Scholes

Evolution of Money Market Account: $\mathrm{d}B = r \cdot B \, \mathrm{d}t$

Evolution of Stock ....: $\mathrm{d}S = \mu \cdot S \mathrm{d}t + \sigma \cdot S \mathrm{d}W(t)$

Choose Numéraire and consider Martingale Measure:

$$B \text{ Numéraire} \;\Rightarrow\; \frac{S}{B} \;\mathbb{Q}\text{-martingale} \;\Rightarrow\; \mu = r$$

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for $\omega = \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \ldots, \omega_N$ (some 1000's samples).

Calculate Payoff for each Sample Path:

$$V(T_n, \omega_j) \; = \; \max\left(S(T_n, \omega_j) - K, 0\right)$$

Calculate Price:

$$\begin{aligned} V(0) \; &= \; B(0) \cdot \mathrm{E}^{\mathbb{Q}}\left(\frac{V(T_n)}{B(T_n)}\right) \; \approx \; \frac{1}{N} \sum_{j=1}^{N} \frac{V(T_n, \omega_j)}{B(T_n, \omega_j)} \\ &= \; \frac{1}{N} \sum_{j=1}^{N} \frac{\max\left(S(T_n, \omega_j) - K, 0\right)}{B(T_n, \omega_j)} \end{aligned}$$

Exercise: Excel Sheet: Monte-Carlo Simulation with Option Pricing

Monte Carlo Simulation with Option Pricing.xlsx

Model Specification

S(0)	100
r	5%
sigma	10%
delta t	0.01

Product Specification

Maturity:	1.00
Strike:	102

Pricing

Option Value:	5.43
Analytic:	5.57

Used in Black Scholes Formula

d+	0.4639983
d-	36%

	A	B	C	D	E	F	G	H	I	CY	DA
10		t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	1.00	
11	i	B(t)	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.05	Option Payout
12	1	S(t,omega_i)	100.00	99.25	99.76	99.15	98.00	98.25	98.41	111.47	9.0119655
13	2	S(t,omega_i)	100.00	98.20	97.81	97.25	95.89	94.17	93.32	95.23	0
14	3	S(t,omega_i)	100.00	97.59	97.13	98.65	98.30	99.18	99.30	107.88	5.5891439
15	4	S(t,omega_i)	100.00	99.90	98.84	98.01	97.50	98.35	98.68	109.60	7.2339195
16	5	S(t,omega_i)	100.00	99.73	99.30	99.97	98.86	100.82	99.80	114.37	11.770016
17	6	S(t,omega_i)	100.00	99.44	98.72	100.96	102.74	101.97	102.57	99.23	0
18	7	S(t,omega_i)	100.00	99.94	101.65	100.07	98.35	98.14	98.36	103.04	0.9931257
19	8	S(t,omega_i)	100.00	101.38	102.30	102.02	103.65	104.76	106.54	95.91	0
20	9	S(t,omega_i)	100.00	100.75	100.61	101.25	101.20	102.54	101.93	113.39	10.8317
21	10	S(t,omega_i)	100.00	100.57	100.42	99.87	99.73	99.00	98.09	99.76	0
22	11	S(t,omega_i)						99.18	98.59	113.73	11.157516
23	12	S(t,omega_i)						105.03	104.02	100.42	0
24	13	S(t,omega_i)						101.42	100.51	112.05	9.5558976
25	14	S(t,omega_i)						99.68	98.56	103.30	1.2364622
26	15	S(t,omega_i)						97.85	97.39	93.99	0
27	16	S(t,omega_i)						102.57	104.10	122.03	19.052697
28	17	S(t,omega_i)						99.73	99.61	97.51	0
29	18	S(t,omega_i)						106.02	107.52	112.19	9.6964967
30	19	S(t,omega_i)						100.75	101.20	100.65	0
31	20	S(t,omega_i)						99.68	100.15	119.47	16.615805
32	21	S(t,omega_i)						95.64	95.04	93.55	0
33	22	S(t,omega_i)						98.53	99.95	105.17	3.016866
34	23	S(t,omega_i)						98.15	95.86	117.40	14.651729
35	24	S(t,omega_i)						100.99	100.78	105.00	2.8548299
36	25	S(t,omega_i)						98.22	99.70	116.82	14.101297
37	26	S(t,omega_i)						102.13	100.89	103.42	1.3537122
38	27	S(t,omega_i)						102.43	102.15	90.90	0
39	28	S(t,omega_i)						99.41	100.20	107.64	5.363011
40	29	S(t,omega_i)	100.00	98.99	97.64	98.76	98.69	98.01	98.19	119.96	17.084204
41	30	S(t,omega_i)	100.00	99.10	99.58	97.24	98.75	99.46	98.99	97.53	0
42	31	S(t,omega_i)	100.00	99.47	99.78	100.19	99.21	97.97	98.65	95.82	0
43	32	S(t,omega_i)	100.00	100.07	102.13	102.54	103.45	103.54	104.96	106.44	4.2234097
44	33	S(t,omega_i)	100.00	98.02	99.20	100.96	103.45	105.23	105.57	115.92	13.240557
45	34	S(t,omega_i)	100.00	100.19	100.63	99.53	99.50	99.91	99.63	104.65	2.5222927
46	35	S(t,omega_i)	100.00	100.60	100.63	101.01	102.48	102.35	102.94	93.12	0

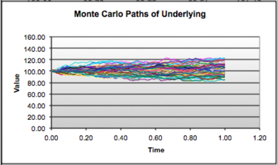

Tabelle1

Normalansicht Bereit Summe= 100

Further Reading I

FRIES, CHRISTIAN P.: Mathematical Finance. Theory, Modeling, Implementation. John Wiley & Sons, 2007. ISBN 0-470-04722-4. http://www.christian-fries.de/finmath/book.