Proxy Simulation Schemes

for Generic Robust Monte-Carlo Sensitivities, Process Oriented Importance Sampling and High Accuracy Drift Approximation

with Applications to the LIBOR Market Model

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Monte Carlo Method

A Short Review of Challenges and Solutions

Discretization Error

Drift Approximations

Monte Carlo Method: Discretization Error

Consider for SDE

$$\mathrm{d}X(t) = \mu(t, X(t))X(t)\mathrm{d}t + \sigma(t, X(t))X(t)\mathrm{d}W(t),$$

e.g. the Log-Euler Scheme

$$X(t + \Delta t) = X(t) \cdot \exp\left(\mu(t, X(t))\Delta t - \frac{1}{2}\sigma^2(t, X(t))\Delta t + \sigma(t, X(t))\Delta W(t)\right).$$

If σ is constant on $[t, \Delta t]$ (Black Model, LIBOR Market Model) but μ is stochastic and/or non-linear (LIBOR Market Model), then the discretization error is given by a drift approximation error, e.g. here

$$\int_t^{t+\Delta t} \mu(\tau, X(\tau))\mathrm{d}\tau \approx \mu(t, X(t))\Delta t.$$

Solutions

- Predictor Corrector Method(s) (= alternative integration rule)
- Proxy Simulation Scheme / Weak Scheme (discussed later)

Sensitivities in Monte Carlo

Partial Derivative with respect to Model Parameters

Monte Carlo Method: Sensitivities

Sensitivities: Let $f(Y)$ denote a random variable depending on realizations $Y := (X(T_1), \ldots X(T_m))$ of our simulated (Numéraire relative) state variables

$$f(Y) \ = \ f(X(T_1), \ldots X(T_m))$$

e.g. the Numéraire relative path values of a financial product. Then the (Numéraire relative) price is given by

$$\mathrm{E}^{\mathbb{Q}}(f(Y) \mid \mathcal{F}_{T_0}) = \mathrm{E}^{\mathbb{Q}}(f(X(T_1), \ldots X(T_m)) \mid \mathcal{F}_{T_0}).$$

Challenge: Let θ denote a parameter of the model SDE (e.g. its initial condition $X(0)$, volatility σ or any other complex function of those). We denote the dependence of the model realizations on θ by Y_θ. We are interested in

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y_\theta) \mid \mathcal{F}_{T_0}) &= \frac{\partial}{\partial\theta}\int_\Omega f(X(T_1,\omega,\theta), \ldots X(T_m,\omega,\theta)) \,\mathrm{d}\mathbb{Q}(\omega) \\ &= \frac{\partial}{\partial\theta}\int_{\mathrm{IR}^m} \underbrace{f(x_1, \ldots x_m)}_{\substack{\text{payoff} \\ \text{may be} \\ \text{discontinuouse}}} \cdot \underbrace{\phi_{(X(T_1,\omega,\theta),\ldots X(T_m,\omega,\theta))}(x_1, \ldots x_m)}_{\text{density - in general smooth in } \theta} \,\mathrm{d}(x_1, \ldots x_m)\end{aligned}$$

Problem: Monte-Carlo approximation inherits regularity of f not of ϕ:

$$\mathrm{E}^{\mathbb{Q}}(Y_\theta \mid \mathcal{F}_{T_0}) \ \approx \ \hat{\mathrm{E}}^{\mathbb{Q}}(Y_\theta \mid \mathcal{F}_{T_0}) \ := \ \frac{1}{n}\sum_{i=1}^{n} \underbrace{f(X(T_1,\omega_i,\theta), \ldots X(T_m,\omega_i,\theta))}_{\substack{\text{payoff on path - may be} \\ \text{discontinuouse}}}$$

Example: Linear and Discontinuous Payouts

Example: Linear and Discontinuous Payout

Linear Payout: First consider a linear payout, say

$$f(X(T)) = a \cdot X(T) + b.$$

Let $Y_\theta(\omega) := X(T, \omega, \theta)$, where θ denotes some model parameter. The partial derivative of the Monte-Carlo value of the payout with respect to θ is

$$\frac{\partial}{\partial\theta}\hat{\mathrm{E}}(f(Y_\theta)|\mathcal{F}_{T_0}) \;=\; \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}f(Y_\theta(\omega_i)) \;=\; \frac{1}{n}\sum_{i=1}^{n} a\cdot\frac{\partial}{\partial\theta}Y_\theta(\omega_i))$$

For the case, when $\frac{\partial}{\partial\theta}Y_\theta(\omega_i)$ does not depend on ω_i, then the Monte-Carlo approximation gives the exact value of the partial derivative, even if we use only a single path.

Finite Differences applied to the Expectation of a Linear Payout

We consider a finite difference approximation of the partial derivative for the case of the linear payout $f(X(T)) = a \cdot X(T) + b$. We have

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y_\theta)\mid\mathcal{F}_{T_0}) \;\approx\; \frac{1}{2h}(\mathrm{E}^{\mathbb{Q}}(f(Y_{\theta+h})\mid\mathcal{F}_{T_0}) - \mathrm{E}^{\mathbb{Q}}(f(Y_{\theta-h})\mid\mathcal{F}_{T_0}))\\
&\approx\; \frac{1}{2h}(\hat{\mathrm{E}}^{\mathbb{Q}}(f(Y_{\theta+h})\mid\mathcal{F}_{T_0}) - \hat{\mathrm{E}}^{\mathbb{Q}}(f(Y_{\theta-h})\mid\mathcal{F}_{T_0})) \;=\; \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2h}(f(Y_{\theta+h}(\omega_i) - f(Y_{\theta-h}(\omega_i))\\
&=\; \frac{1}{n}\sum_{i=1}^{n} a\cdot\frac{1}{2h}(Y_{\theta+h}(\omega_i) - Y_{\theta-h}(\omega_i)),
\end{aligned}$$

which is a good approximation if the model realizations $Y_\theta(\omega_i)$ are smooth in θ.

Example: Linear and Discontinuous Payout

Discontinuous Payout: Next, consider a discontinuous payout, say

$$f(X(T)) = \begin{cases} 1 & \text{if } X(T) > K \\ 0 & \text{else.} \end{cases}$$

Analytically we have from $Y_{\theta+h} = Y_\theta + \frac{\partial Y_\theta}{\partial \theta} \cdot h \ + O(h^2)$ and

$$\mathrm{E}^{\mathbb{Q}}(f(Y_{\theta+h}) \mid \mathcal{F}_{T_0}) \ = \ \mathbb{Q}(\{Y_\theta > K - \frac{\partial Y_\theta}{\partial \theta} \cdot h - O(h^2)\}) \ = \ \int_{K - \frac{\partial Y_\theta}{\partial \theta} \cdot h - O(h^2)}^{\infty} \phi_{Y_\theta}(y) \mathrm{d}y$$

that

$$\lim_{h \to 0} \frac{1}{2h} (\mathrm{E}^{\mathbb{Q}}(f(Y_{\theta+h}) \mid \mathcal{F}_{T_0}) - \mathrm{E}^{\mathbb{Q}}(f(Y_{\theta-h}) \mid \mathcal{F}_{T_0})) \ = \ \phi_{Y_\theta}(K) \cdot \frac{\partial Y_\theta}{\partial \theta}.$$

The partial derivative of the Monte-Carlo value of the payout is

$$\frac{\partial}{\partial \theta} \hat{\mathrm{E}}(f(Y_\theta) | \mathcal{F}_{T_0}) \ = \ \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial \theta} f(Y_\theta(\omega_i)) \ = \ 0 \text{ assuming that } Y_\theta(\omega_i) \neq K \text{ for all } i.$$

Thus, here, the partial derivative of the Monte-Carlo value is always wrong.

Reason:

We are not allow to interchange the order of limits (number of paths $\to \infty$) and (shift size $\to 0$).

Finite Differences applied to the Expectation of a Discontinuous Payout

For the discontinuous payout $f(X(T)) = 1$ if $X(T) > K$ and $f(X(T)) = 0$ else, we have

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y_\theta) \mid \mathcal{F}_{T_0}) &\approx \frac{1}{2h}(\mathrm{E}^{\mathbb{Q}}(f(Y_{\theta+h}) \mid \mathcal{F}_{T_0}) - \mathrm{E}^{\mathbb{Q}}(f(Y_{\theta-h}) \mid \mathcal{F}_{T_0})) \\
&\approx \frac{1}{2h}(\hat{\mathrm{E}}^{\mathbb{Q}}(f(Y_{\theta+h}) \mid \mathcal{F}_{T_0}) - \hat{\mathrm{E}}^{\mathbb{Q}}(f(Y_{\theta-h}) \mid \mathcal{F}_{T_0})) \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2h}(f(Y_{\theta+h}(\omega_i) - f(Y_{\theta-h}(\omega_i)) \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2h}\begin{cases} 1 & \text{if } Y_{\theta-h}(\omega_i) < K < Y_{\theta+h}(\omega_i) \\ -1 & \text{if } Y_{\theta-h}(\omega_i) > K > Y_{\theta+h}(\omega_i) \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

This is a valid approximation, but it has a large Monte-Carlo variance, since the true value is sampled by 0 and $\frac{1}{2h}$ occurring in the appropriate frequency. If h gets smaller then we have to represent true value by a sampling of 0 and a very large constant.

Example: Linear and Discontinuous Payout

Simplified Example: Assume for simplicity that Y_θ is linear in θ, i.e. we have

$$\frac{f(Y_{\theta+h}(\omega_i)) - f(Y_{\theta-h}(\omega_i))}{2h} = \begin{cases} \frac{1}{2h} & \text{if } Y_{\theta-h}(\omega_i) < K < Y_{\theta+h}(\omega_i) \\ \frac{-1}{2h} & \text{if } Y_{\theta-h}(\omega_i) > K > Y_{\theta+h}(\omega_i) \\ 0 & \text{else.} \end{cases} = \begin{cases} \frac{\operatorname{sign} \frac{\partial Y_\theta}{\partial \theta}}{2h} & \text{if } Y_\theta(\omega_i) \in [K-\epsilon, K+\epsilon] \\ 0 & \text{else.} \end{cases}$$

where $\epsilon := \left|\frac{\partial Y_\theta}{\partial \theta}\right| \cdot h$. For the probability we have

$$q \;:=\; \mathbb{Q}(Y_\theta \in [K-\epsilon, K+\epsilon]) \;\approx\; \phi_{Y_\theta}(K) \cdot 2\epsilon \;=\; \phi_{Y_\theta}(K) \cdot \left|\frac{\partial Y_\theta}{\partial \theta}\right| \cdot 2h.$$

In other words: We are sampling the partial derivative of the expectation by a binomial experiment:

$$\frac{\operatorname{sign} \frac{\partial Y_\theta}{\partial \theta}}{2h} \quad \text{with probability } q \qquad \text{and} \qquad 0 \quad \text{with probability } 1-q.$$

The expectation of this binomial experiment is

$$\frac{\operatorname{sign} \frac{\partial Y_\theta}{\partial \theta}}{2h} \cdot q + 0 \cdot (1-q) \;\approx\; \phi_{Y_\theta}(K) \cdot \frac{\partial Y_\theta}{\partial \theta},$$

which is the desired analytic value for the finite difference approximation as $h \to 0$. The variance of the binomial experiment is

$$\left(\frac{1}{2h}\right)^2 \cdot q \cdot (1-q) \;\approx\; \phi_{Y_\theta}(K) \cdot \frac{\partial Y_\theta}{\partial \theta} \cdot (1-q) \cdot \frac{1}{2h} \;=\; O\left(\frac{1}{2h}\right),$$

which explodes as $h \to 0$.

Example: Auto Cap

Example: AutoCap Sensitivities: Cap Products

Caplet: Single option on forward rate. Payoff profile:

$$\max(L_i(T_i) - K, 0) \cdot (T_{i+1} - T_i) \quad \text{paid in } T_{i+1}$$

Cap: Portfolio (series) of n options on forward rates (Caplets). Value = Sum of Caplets.

Chooser Cap: Cap, where only some ($k < n$) options may be exercised. Holder may choose upon each excersie date. Value is given by optimal exercise strategy.
⇒ Value depends continuously on model & product parameters.

Auto Cap: Cap, where only some ($k < n$) options may be exercised. Excercise is triggered is Caplet payout is positive. Payoff profile:

$$\left.\begin{array}{ll} \max(L_i(T_i) - K, 0) \cdot (T_{i+1} - T_i) & \text{if } \left|\left\{ j \ : \ L_j(T_j) - K > 0 \text{ and } j < i \right\}\right| < k \\ 0 & \text{else} \end{array}\right\} \text{paid in } T_{i+1}.$$

Auto Cap Features:

- On a single (fixed) path the product depends discontinuously on the input data (e.g. todays interest rate level). Note: Chooser Cap depends continuously on model & product parameters.

Example: AutoCap Sensitivities

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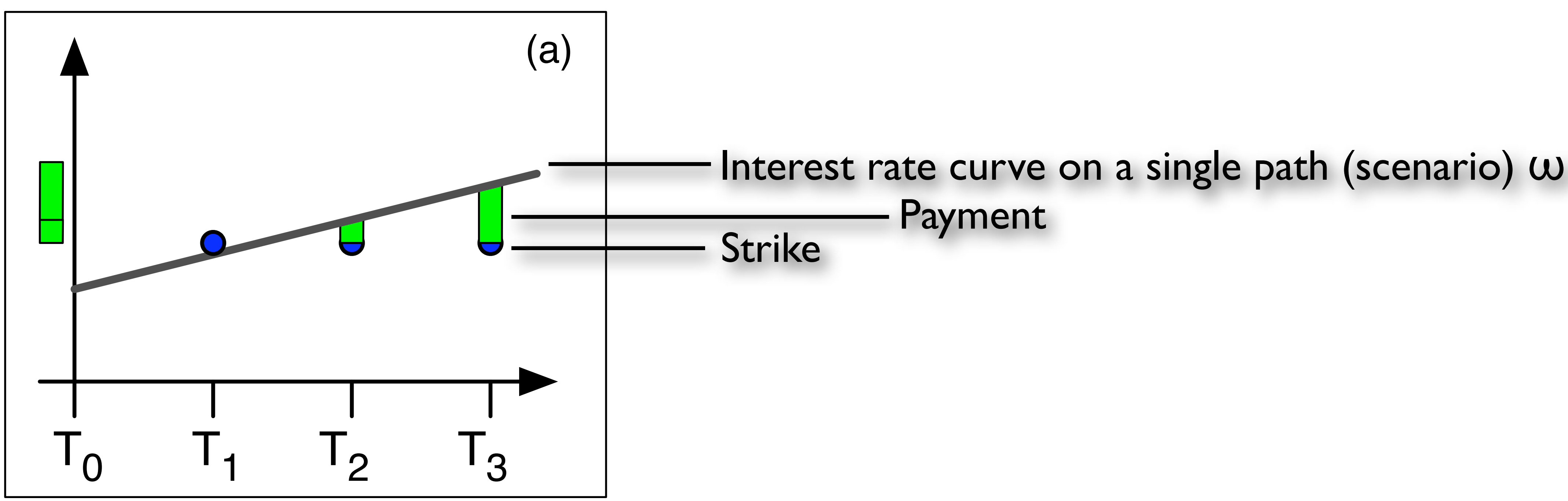
Example: Auto Cap pays first two positive caplet payouts out of three, all strikes the same, upward sloping forward rate curve, first caplet slightly out of the money.

Parallel upshift of the forward rate curve will bring first caplet in the money (small payoff)
$\Rightarrow$ (large) payoff of last caplet is lost.



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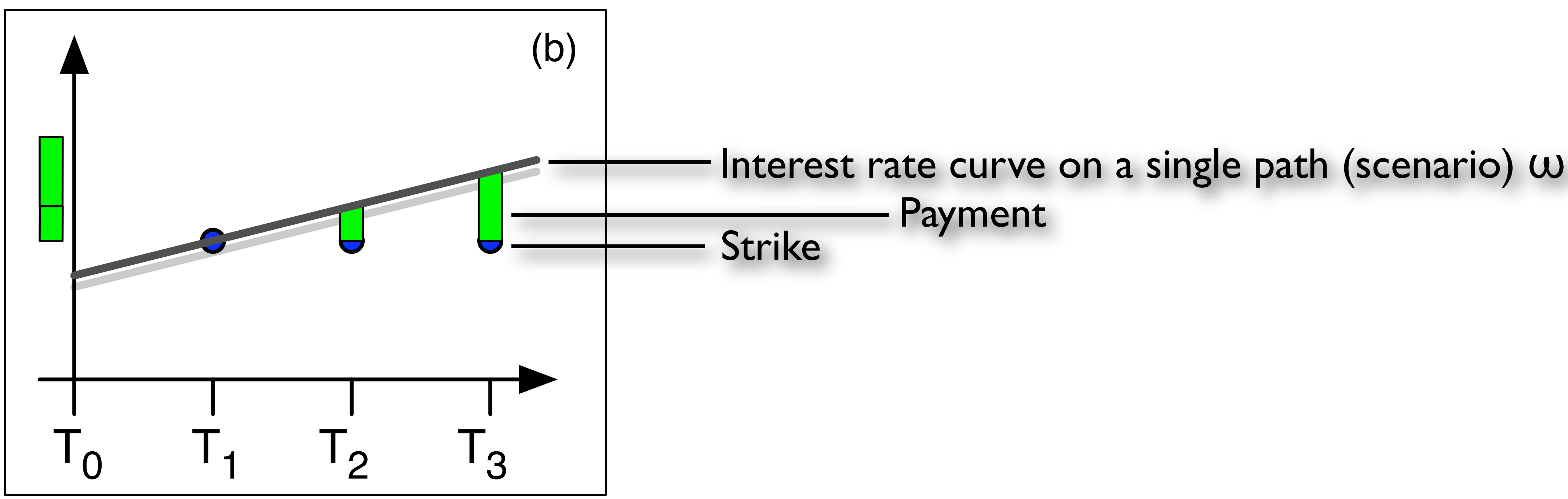

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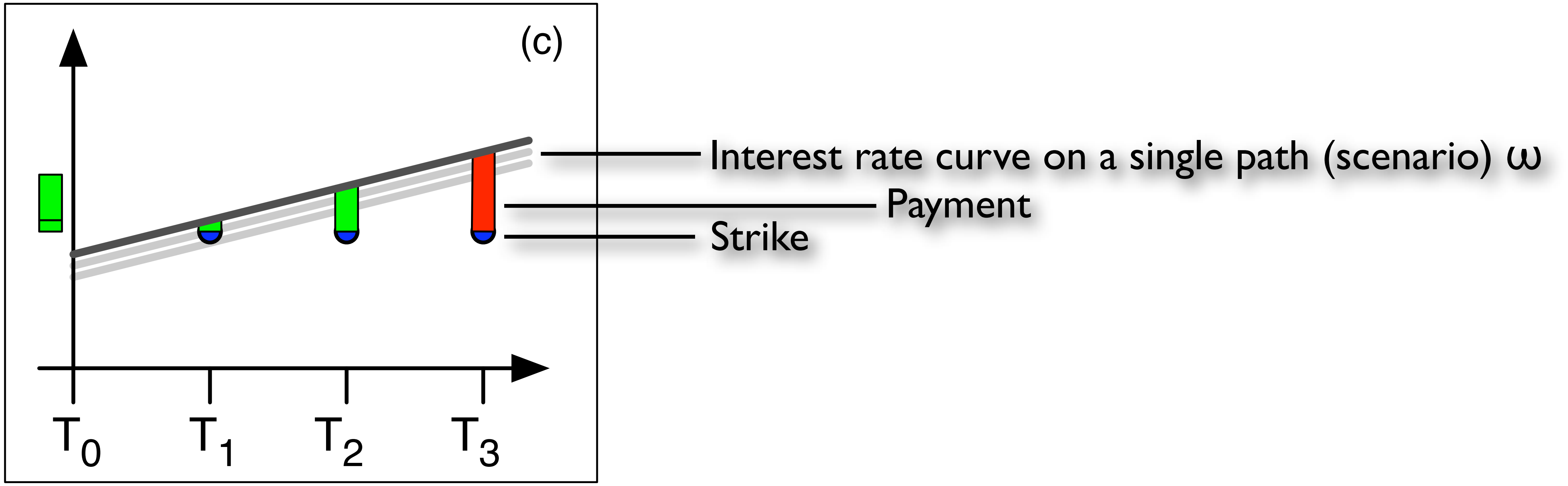

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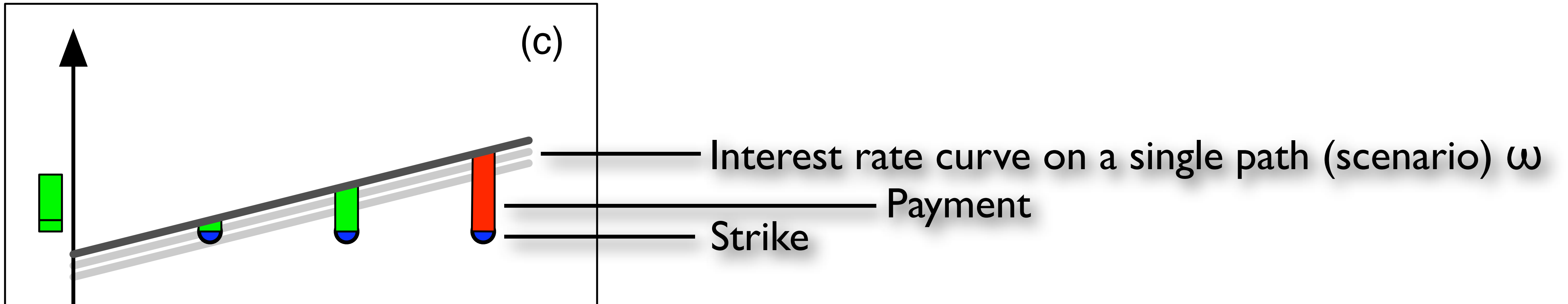
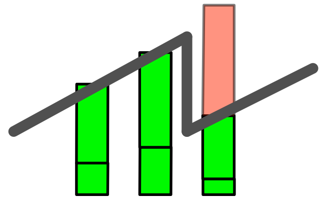
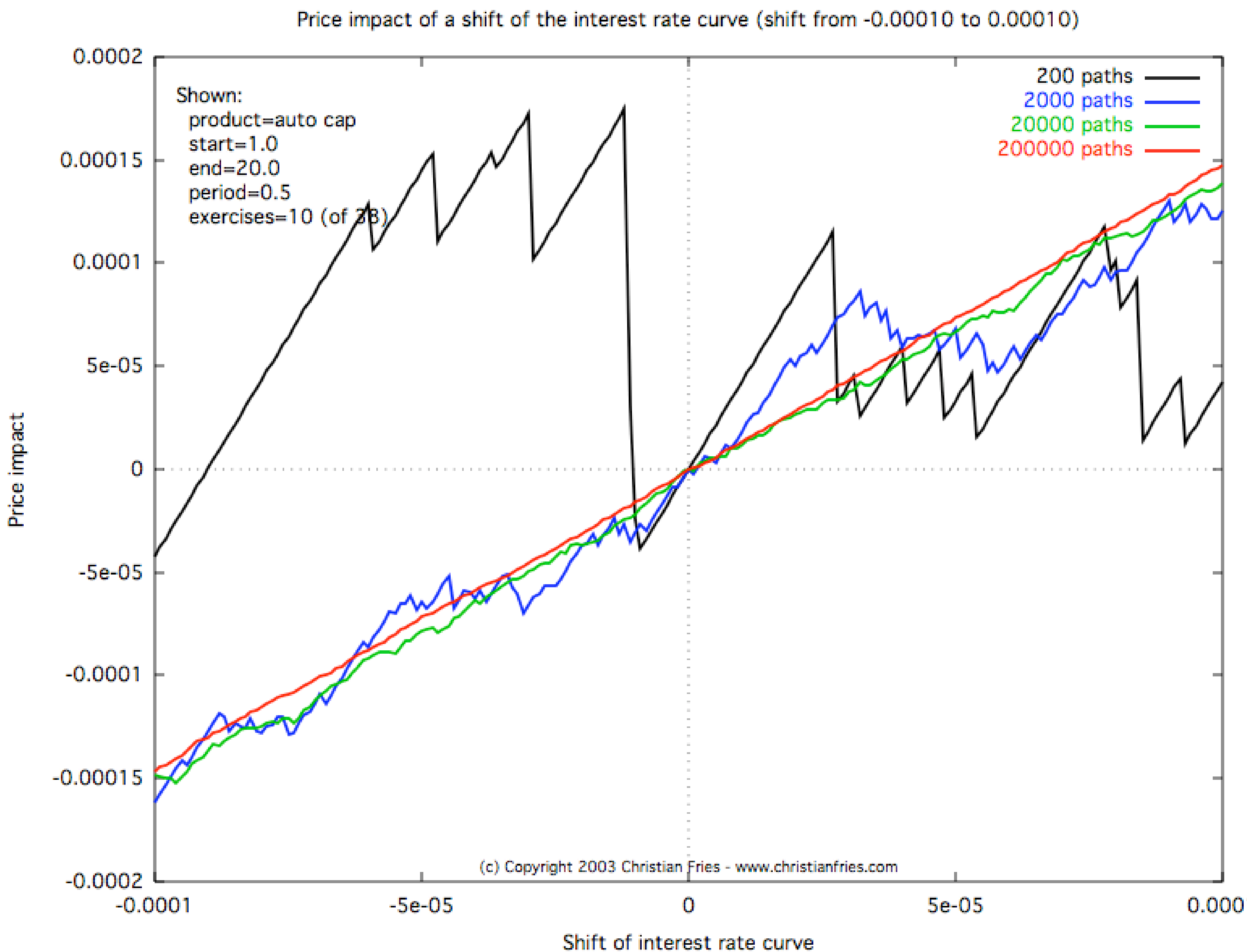




Example: AutoCap Sensitivities: 1 bp shift



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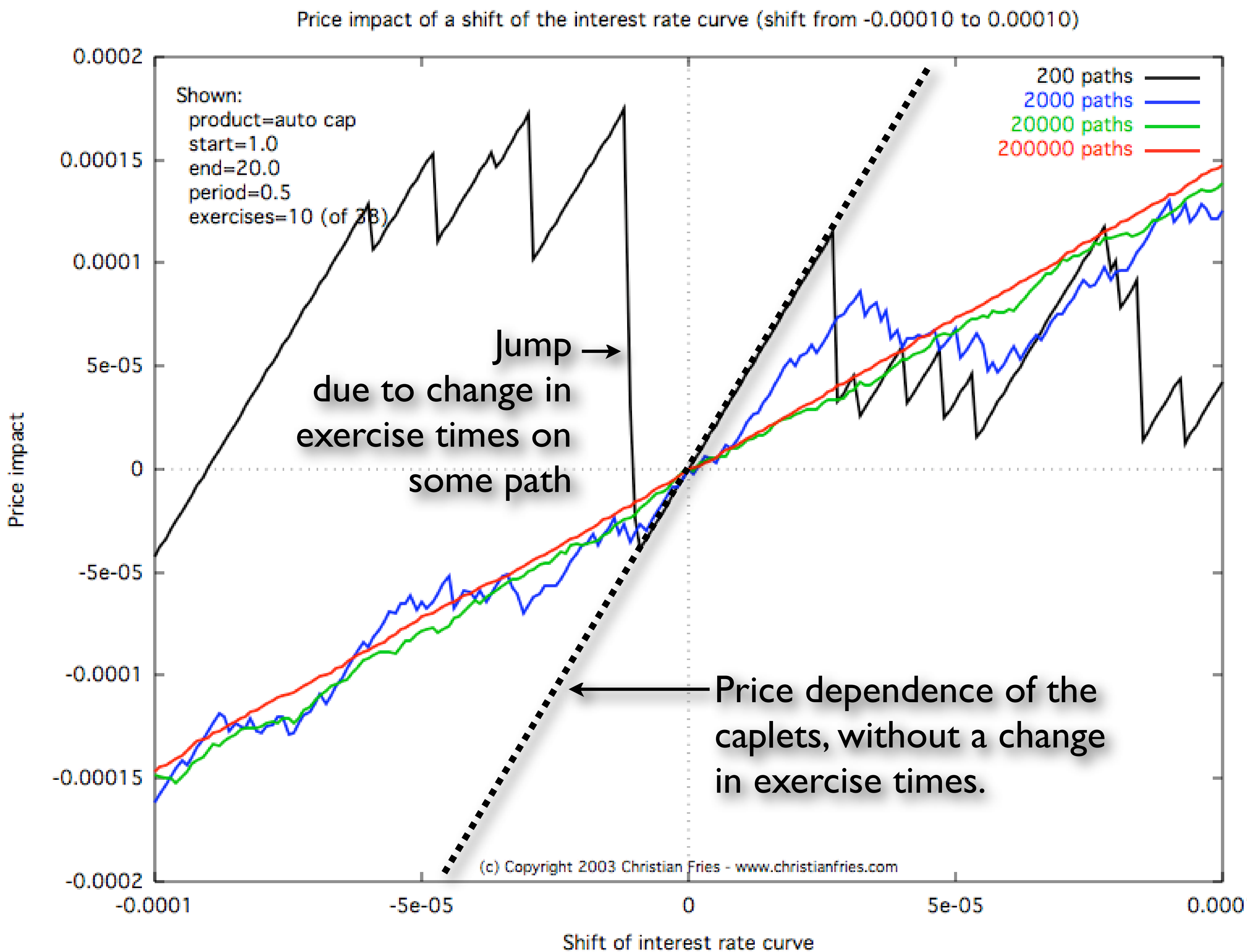

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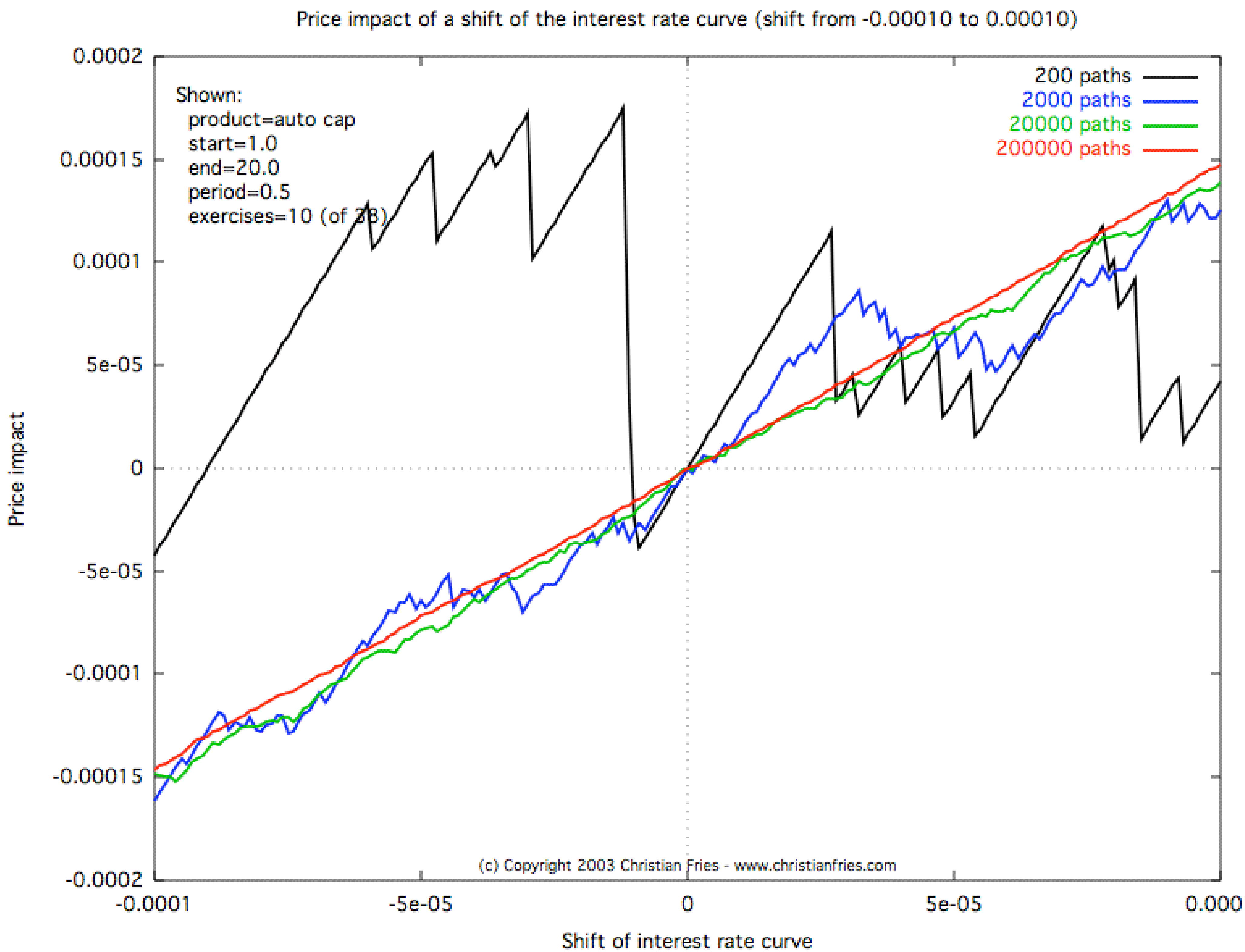
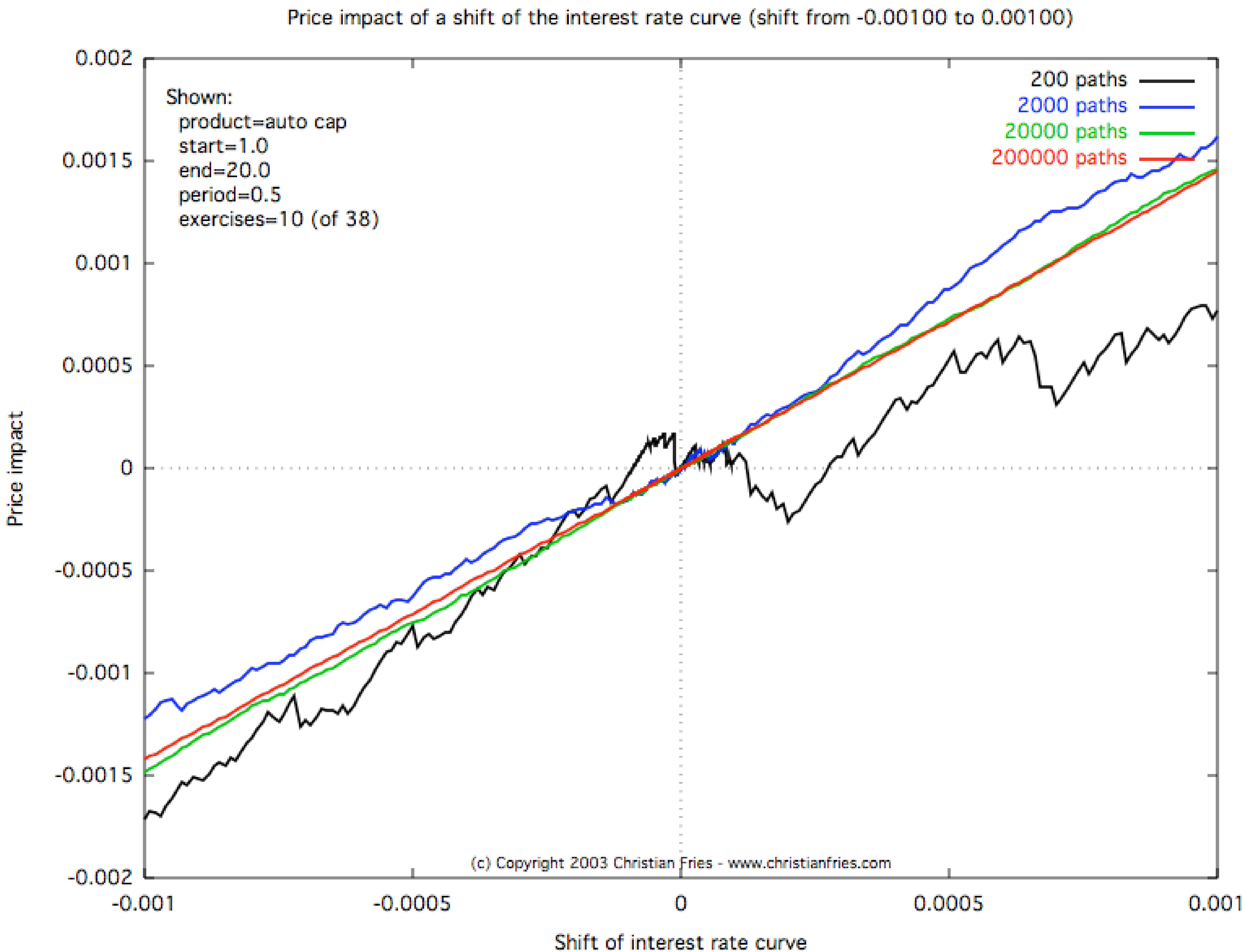


Example: AutoCap Sensitivities: 10 bp shift



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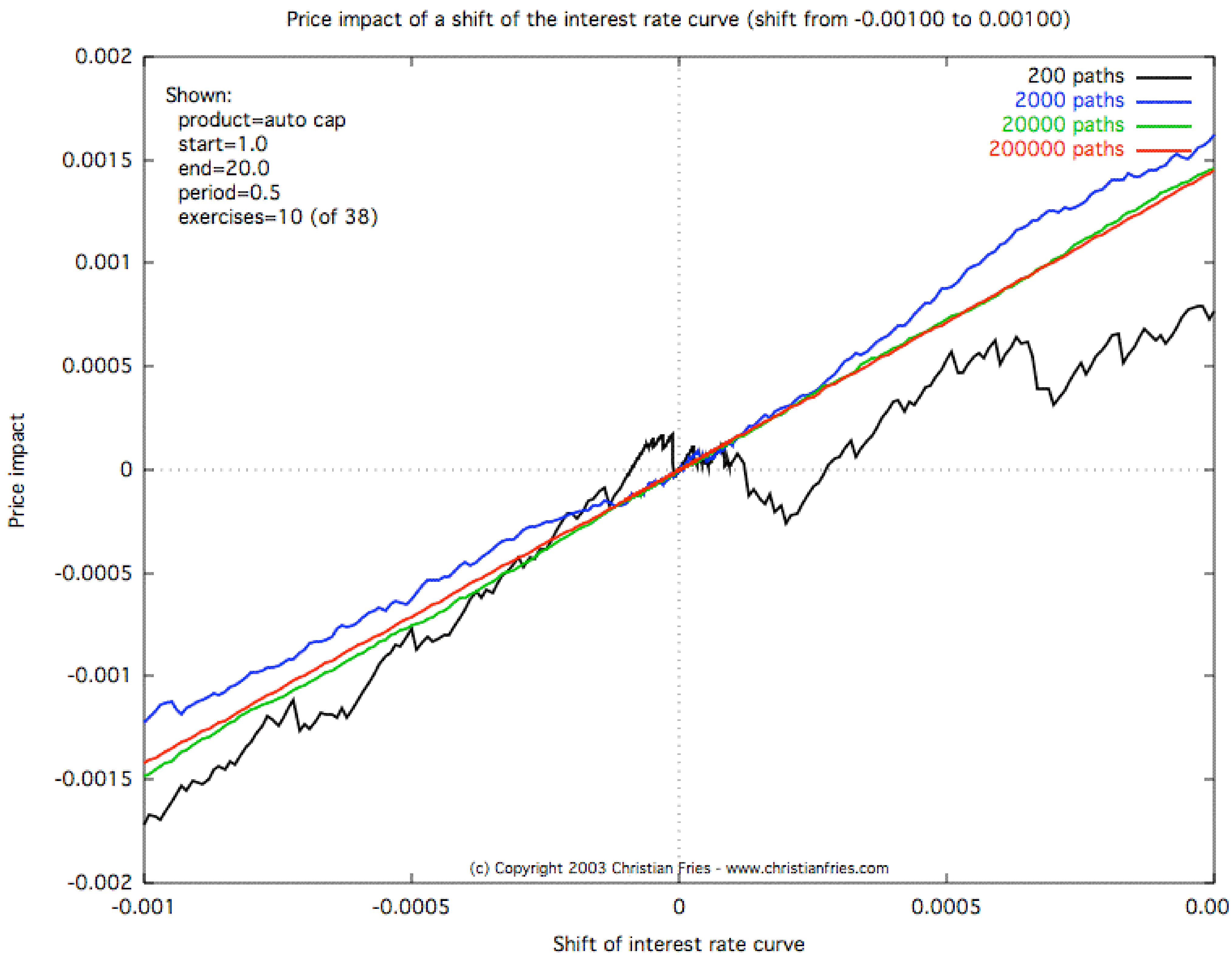
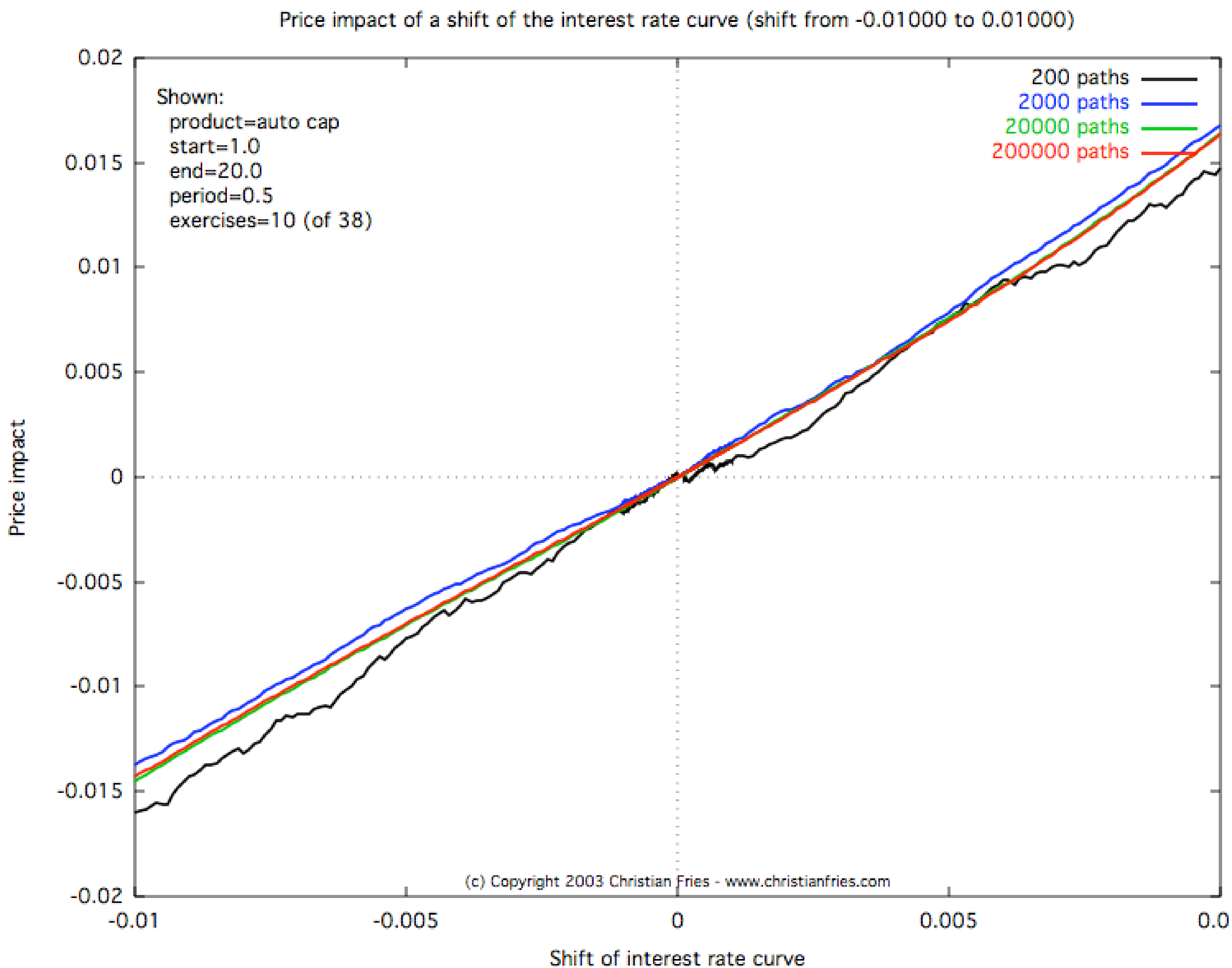
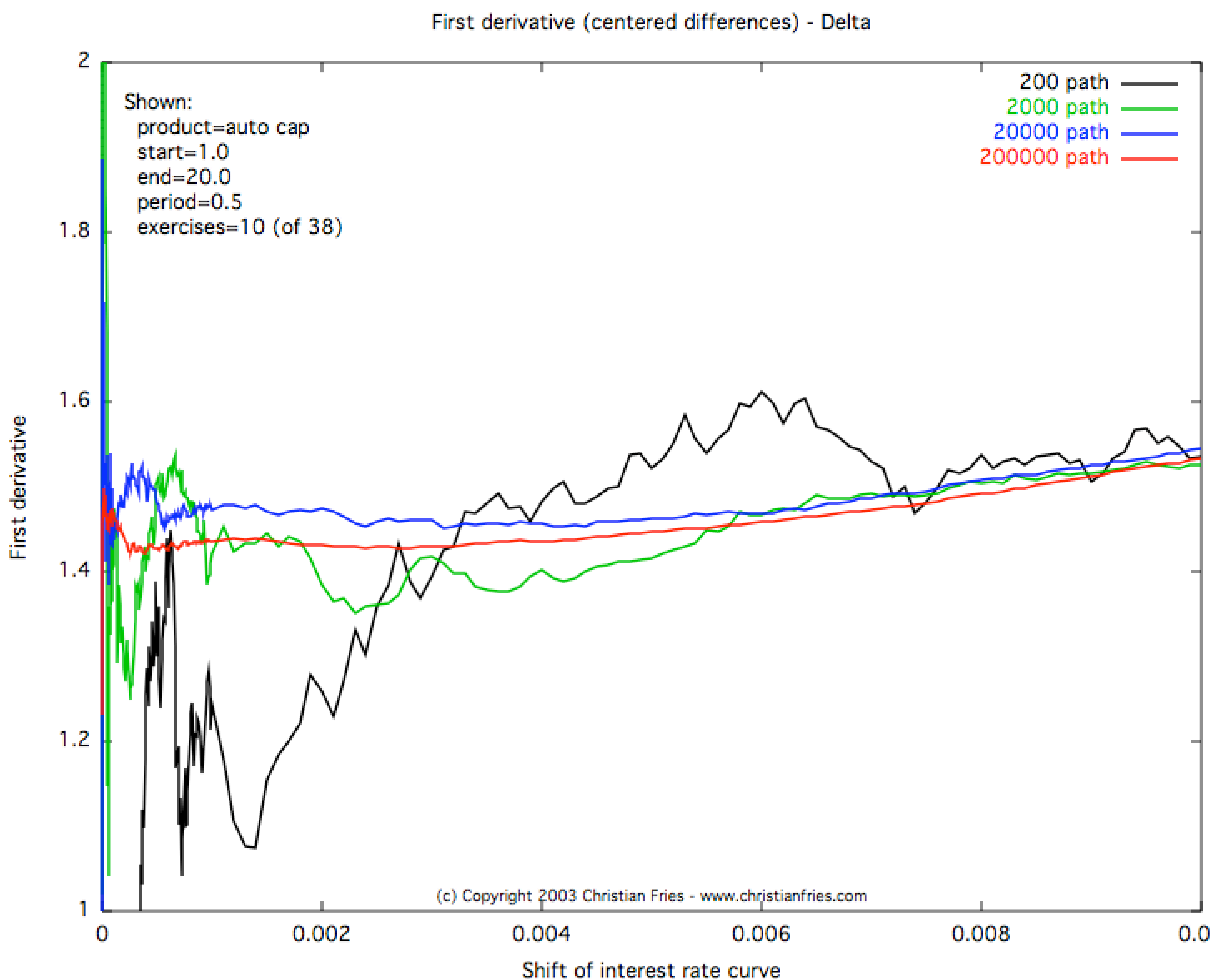


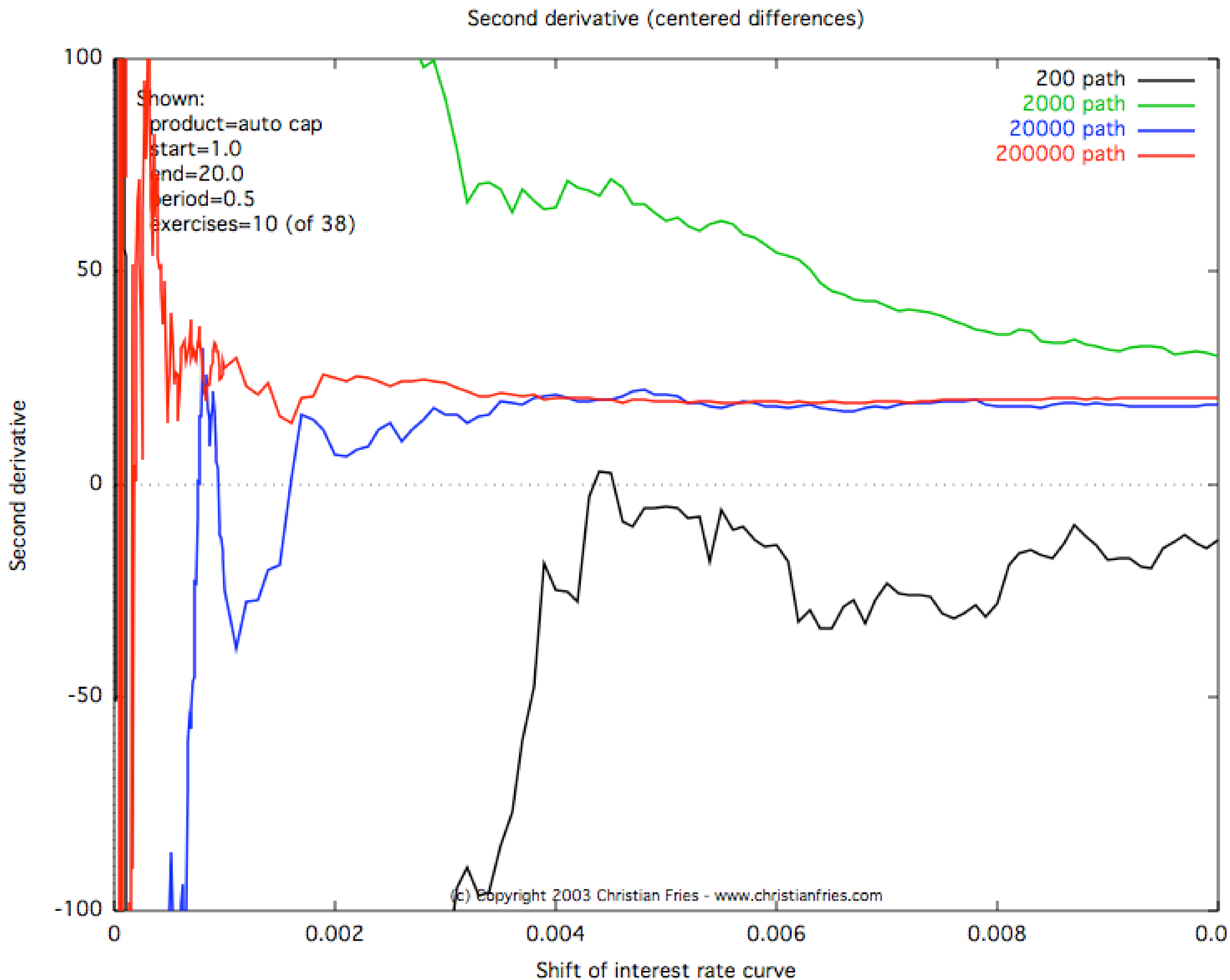
Example: AutoCap Sensitivities: 100 bp shift



Example: AutoCap Sensitivities: Delta 100 bp



Example: AutoCap Sensitivities: Gamma 100 bp



Example: TARN

Example: Target Redemption Notes

TARN: A tarn pays

$$N_i \cdot X_i \qquad \text{in} \qquad T_{i+1},$$

where

$$
\begin{aligned}
X_i := \quad & \min(C_i\,,K - \sum_{k=1}^{i-1} C_k) && \text{(structured coupon)} \\
& + \begin{cases} 1 & \text{for } \sum_{k=1}^{i-1} C_k < K <= \sum_{k=1}^{i} C_k \text{ or } i = n, \\ 0 & \text{else.} \end{cases} && \text{(redemption)} \\
& + \begin{cases} \max(0\,,\; K - \sum_{k=1}^{i} C_k) & \text{for } i = n \\ 0 & \text{else.} \end{cases} && \text{(target coupon guarantee).}
\end{aligned}
$$

TARN Features:

- On a single (fixed) path the product depends discontinuously on the input data (e.g. todays interest rate level). Here it is the timing of the redemption payment that constitutes the discontinuity. It is triggered by the sum of the coupons C_i.

Generic Sensitivites

Bumping the Model

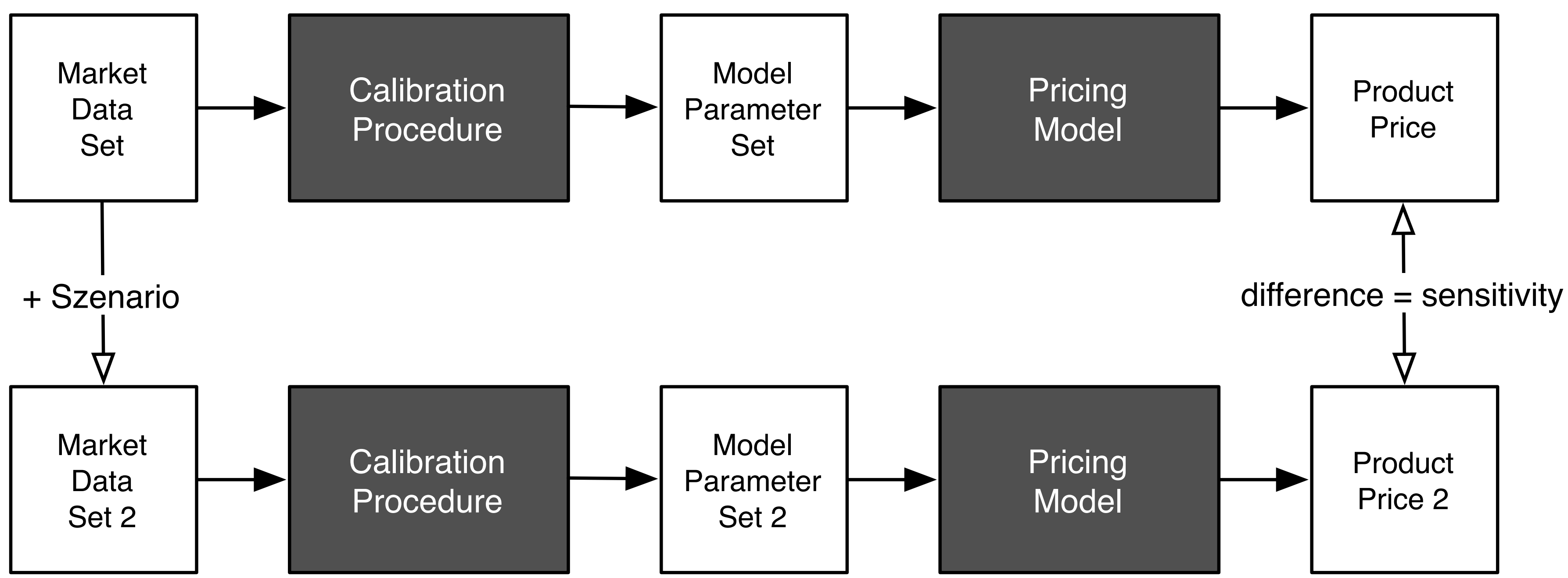
A Note on *Generic Sensitivities*

What a mathematician considers as the "delta" of an option is not what a trader considers as the "delta".

After a change in market data a model has to be recalibrated.

Example: Given the assumption of a certain volatility modeling (e.g. sticky strike versus sticky moneyness), a change in the underlying might also imply a change in the whole volatility surface.

We have to distinguish *(generic) market sensitivities* and *model sensitivities*.



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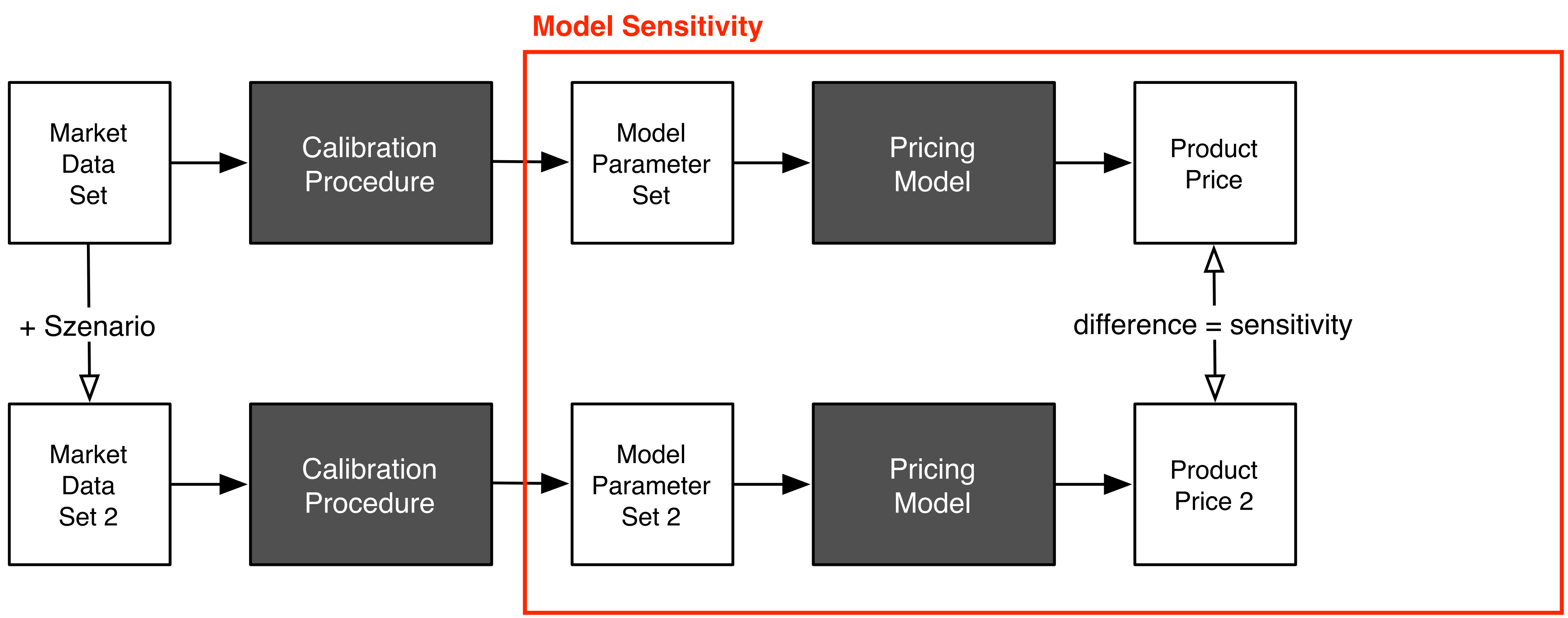

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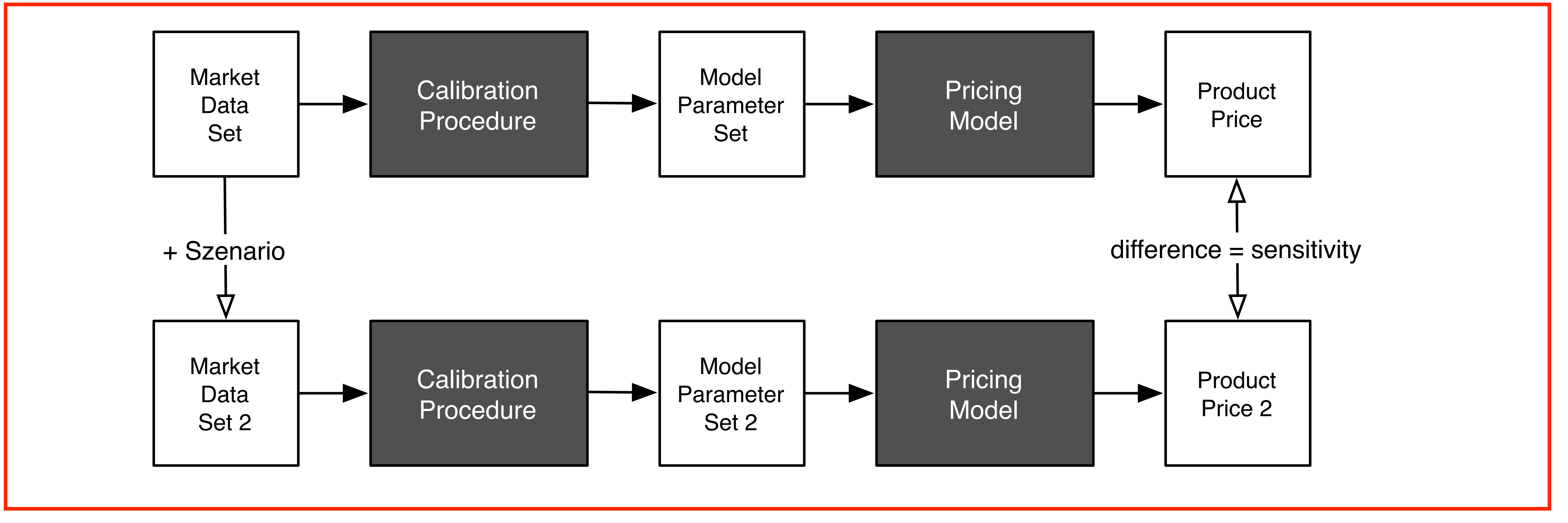
Market Sensitivity



A Note on *Generic Sensitivities*

Methods for calculating generic sensitivities:

- Finite Differences
  Problem: May be numerically unstable.

- Chain rule and
  - finite differences for market data / calibration
  - some other method (see below) for model sensitivities

  Problem: May require full set of model sensitivities.

- Finite Differences on a Proxy Simulation Scheme

Methods for calculating model sensitivities:

- Finite Differences
- Pathwise Differentiation
- Likelihood Ration Method
- Malliavin Calculus

Sensitivities in Monte Carlo

Overview

Monte Carlo Methods: Sensitivities

Finite Differences:

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y_\theta) \mid \mathcal{F}_{T_0}) &\approx \frac{1}{2h}(\mathrm{E}^{\mathbb{Q}}(f(Y_{\theta+h}) \mid \mathcal{F}_{T_0}) - \mathrm{E}^{\mathbb{Q}}(f(Y_{\theta-h}) \mid \mathcal{F}_{T_0})) \\
&\approx \frac{1}{2h}(\hat{\mathrm{E}}^{\mathbb{Q}}(f(Y_{\theta+h}) \mid \mathcal{F}_{T_0}) - \hat{\mathrm{E}}^{\mathbb{Q}}(f(Y_{\theta-h}) \mid \mathcal{F}_{T_0})) \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2h}(f(Y_{\theta+h}(\omega_i)) - f(Y_{\theta-h}(\omega_i)))
\end{aligned}
$$

Requirements

- Requires no additional information from the model sde $\mathrm{d}X = \ldots$
- Requires no additional information from the simulation scheme $X(T_{i+1}) = \ldots$
- Requires no additional information from the payout f
- Requires no additional information on the nature of θ ($\Rightarrow$ generic sensitivities)

Properties

- **Generic sensitivities (market sensitivies)**
- Biased derivative for *large* h due to finite difference of order h
- Large variance for discontinuous payouts and *small* h (order h^{-1})

Monte Carlo Methods: Sensitivities: Pathwise Differentiation

Pathwise Differentiation:

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y(\theta)) \mid \mathcal{F}_{T_0}) &= \frac{\partial}{\partial\theta}\int_\Omega f(Y(\omega,\theta))\,\mathrm{d}\mathbb{Q}(\omega) = \int_\Omega \frac{\partial}{\partial\theta} f(Y(\omega,\theta))\,\mathrm{d}\mathbb{Q}(\omega) \\
&= \int_\Omega f'(Y(\omega,\theta)) \cdot \frac{\partial Y(\omega,\theta)}{\partial\theta}\,\mathrm{d}\mathbb{Q}(\omega) = \mathrm{E}^{\mathbb{Q}}(f'(Y(\theta)) \cdot \frac{\partial Y(\theta)}{\partial\theta} \mid \mathcal{F}_{T_0}) \\
&\overset{f \text{ smooth}}{\approx} \hat{\mathrm{E}}^{\mathbb{Q}}(f'(Y(\theta)) \cdot \frac{\partial Y(\theta)}{\partial\theta} \mid \mathcal{F}_{T_0}) = \frac{1}{n}\sum_{i=1}^{n} f'(Y(\omega_i,\theta)) \cdot \frac{\partial Y(\omega_i,\theta)}{\partial\theta}
\end{aligned}$$

Requirements

- Requires additional information on the model sde $\mathrm{d}X = \ldots$
- Requires no additional information on the simulation scheme $X(T_{i+1}) = \ldots$
- Requires additional information on the payout f (derivative of f must be known)
- Requires additional information on the nature of θ ($\Rightarrow$ restricted class of model parameters)

Properties

- **No generic gensitivities (model sensitivies only)**
- Unbiased derivative
- Requires smoothness of payout? (in this formulation)

Pathwise Differentiation:

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y(\theta)) \mid \mathcal{F}_{T_0}) &= \frac{\partial}{\partial\theta}\int_\Omega f(Y(\omega,\theta))\,\mathrm{d}\mathbb{Q}(\omega) = \int_\Omega \frac{\partial}{\partial\theta} f(Y(\omega,\theta))\,\mathrm{d}\mathbb{Q}(\omega) \\
&= \int_\Omega f'(Y(\omega,\theta)) \cdot \frac{\partial Y(\omega,\theta)}{\partial\theta}\,\mathrm{d}\mathbb{Q}(\omega) = \mathrm{E}^{\mathbb{Q}}(f'(Y(\theta)) \cdot \frac{\partial Y(\theta)}{\partial\theta} \mid \mathcal{F}_{T_0}) \\
&\overset{g \text{ smooth}}{\approx} \hat{\mathrm{E}}^{\mathbb{Q}}(g'(Y(\theta)) \cdot \frac{\partial Y(\theta)}{\partial\theta} \mid \mathcal{F}_{T_0}) + \sum_i \alpha_i \cdot \phi(y_i) \cdot \frac{\partial Y(\theta)}{\partial\theta}\Big|_{Y(\theta)=y_i}
\end{aligned}$$

Where $f = g + \sum_i \alpha_i \mathbf{1}(\{Y(\theta) > y_i\})$. See Joshi & Kainth [JK] or Rott & Fries [RF] for an example on how use pathwise differentiation with discontinuous payouts (there in the context of Default Swaps, CDOs).

Requirements

- Requires additional information on the model sde $\mathrm{d}X = \ldots$
- Requires no additional information on the simulation scheme $X(T_{i+1}) = \ldots$
- Requires additional information on the payout f (derivative of f must be known)
- Requires additional information on the nature of θ ($\Rightarrow$ restricted class of model parameters)

Properties

- **No generic gensitivities (model sensitivies only)**
- Unbiased derivative
- Discontinuous payouts may be handled (interpret f' as distribution, for applications see e.g. [JK, RF])

Likelihood Ratio:

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y(\theta)) \mid \mathcal{F}_{T_0}) &= \frac{\partial}{\partial\theta}\int_{\Omega} f(Y(\omega,\theta))\, \mathrm{d}\mathbb{Q}(\omega) = \frac{\partial}{\partial\theta}\int_{\mathbb{R}^m} f(y)\cdot\phi_{Y(\theta)}(y)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^m} f(y)\cdot\frac{\frac{\partial}{\partial\theta}\phi_{Y(\theta)}(y)}{\phi_{Y(\theta)}(y)}\cdot\phi_{Y(\theta)}(y)\, \mathrm{d}y = \mathrm{E}^{\mathbb{Q}}(f(Y)\cdot w(\theta) \mid \mathcal{F}_{T_0}) \\ &\approx \hat{\mathrm{E}}^{\mathbb{Q}}(f(Y)\cdot w(\theta) \mid \mathcal{F}_{T_0}) = \frac{1}{n}\sum_{i=1}^{n} f(Y(\omega_i))\cdot w(\theta,\omega_i)\end{aligned}$$

Requirements

- Requires additional information on the model sde $\mathrm{d}X = \ldots$ ($\rightarrow \phi_{Y(\theta)}$)
- Requires no additional information on the simulation scheme $X(T_{i+1}) = \ldots$
- Requires no additional information on the payout f
- Requires additional information on the nature of θ ($\Rightarrow$ restricted class of model parameters)

Properties

- **No generic gensitivities (model sensitivies only)**
- Unbiased derivative
- Discontinuous payouts may be handled, but large variance for smooth payouts.

Malliavin Calculus:

$$\begin{aligned}\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y(\theta)) \mid \mathcal{F}_{T_0}) &= \mathrm{E}^{\mathbb{Q}}(f(Y(\theta)) \cdot w(\theta) \mid \mathcal{F}_{T_0}) \\ &\approx \hat{\mathrm{E}}^{\mathbb{Q}}(f(Y(\theta)) \cdot w(\theta) \mid \mathcal{F}_{T_0}) = \frac{1}{n}\sum_{i=1}^{n} f(Y(\theta,\omega_i)) \cdot w(\theta,\omega_i)\end{aligned}$$

Note: Benhamou [B01] showed that the Likelihood Ratio corresponds to the Malliavin weights with minimal variance and may be expressed as a conditional expectation of all corresponding Malliavin weights (we thus view the Likelihood Ratio as an example for the Malliavin weighting method).

Requirements

- Requires additional information on the model sde $\mathrm{d}X = \ldots$ ($\rightarrow w$)
- Requires no additional information on the simulation scheme $X(T_{i+1}) = \ldots$
- Requires no additional information on the payout f
- Requires additional information on the nature of θ ($\Rightarrow$ restricted class of model parameters)

Properties

- **No generic gensitivities (model sensitivies only)**
- Unbiased derivative
- Discontinuous payouts may be handled, but large variance for smooth payouts.

Proxy Simulation Scheme

Proxy Simulation Scheme: Definition / Pricing

Proxy Scheme Simulation: Pricing

Proxy Scheme: Consider *two* numerical schemes

X	$t \mapsto X(t)$	$t \in \mathbb{R}$	model sde
X^*	$T_i \mapsto X^*(T_i)$	$i = 0, 1, 2, \ldots$	time discretization scheme of X → *target scheme*
X°	$T_i \mapsto X^\circ(T_i)$	$i = 0, 1, 2, \ldots$	any other time discrete stochastic process (assumed to be *close* to X^*) → *proxy scheme*

Pricing: Let $Y = (X(T_1), \ldots, X(T_m))$, $Y^* = (X^*(T_1), \ldots, X^*(T_m))$, $Y^\circ = (X^\circ(T_1), \ldots, X^\circ(T_m))$.
We have $\mathrm{E}^{\mathbb{Q}}(f(Y(\theta)) \mid \mathcal{F}_{T_0}) \approx \mathrm{E}^{\mathbb{Q}}(f(Y^*(\theta)) \mid \mathcal{F}_{T_0})$ and furthermore

$$\begin{aligned}
\mathrm{E}^{\mathbb{Q}}(f(Y^*(\theta)) \mid \mathcal{F}_{T_0}) &= \int_\Omega f(Y^*(\omega, \theta))\, \mathrm{d}\mathbb{Q}(\omega) = \int_{\mathbb{R}^m} f(y) \cdot \phi_{Y^*(\theta)}(y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^m} f(y) \cdot \frac{\phi_{Y^*(\theta)}(y)}{\phi_{Y^\circ}(y)} \cdot \phi_{Y^\circ}(y)\, \mathrm{d}y = \mathrm{E}^{\mathbb{Q}}(f(Y^\circ) \cdot w(\theta) \mid \mathcal{F}_{T_0})
\end{aligned}$$

where $w(\theta) = \frac{\phi_{Y^*(\theta)}(y)}{\phi_{Y^\circ}(y)}$.

Note:
- For $X^\circ = X^*$ we have $w(\theta) = 1 \Rightarrow$ ordinary Monte Carlo.
- Y° is seen as being independent of θ. $\Rightarrow$ implications on sensitivities.
- Requirement: $\forall y : \phi^{Y^\circ}(y) = 0 \Rightarrow \phi^{Y^*}(y) = 0$

Proxy Simulation Scheme: Sensitivites

Proxy Scheme Simulation: Sensitivities

Proxy Scheme Sensitivities:

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y^{*}(\theta)) \mid \mathcal{F}_{T_0}) &= \frac{\partial}{\partial\theta}\int_{\Omega} f(Y^{*}(\omega,\theta))\,\mathrm{d}\mathbb{Q}(\omega) = \frac{\partial}{\partial\theta}\int_{\mathbb{R}^m} f(y)\cdot\phi_{Y^{*}(\theta)}(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} f(y)\cdot\frac{\frac{\partial}{\partial\theta}\phi_{Y^{*}(\theta)}(y)}{\phi_{Y^{\circ}}(y)}\cdot\phi_{Y^{\circ}}(y)\,\mathrm{d}y = \mathrm{E}^{\mathbb{Q}}(f(Y^{\circ})\cdot w'(\theta) \mid \mathcal{F}_{T_0}) \\
&\approx \hat{\mathrm{E}}^{\mathbb{Q}}(f(Y^{\circ})\cdot w'(\theta) \mid \mathcal{F}_{T_0}) = \frac{1}{n}\sum_{i=1}^{n} f(Y^{\circ}(\omega_i))\cdot w'(\theta,\omega_i)
\end{aligned}$$

Requirements

- Requires no additional information on the model sde $\mathrm{d}X = \ldots$
- Requires additional information on the simulation scheme $X^{*}(T_{i+1})$, $X^{\circ}(T_{i+1})$
- Requires no additional information on the payout f
- Requires additional information on the nature of θ ($\Rightarrow$ restricted class of model parameters)

Properties

- **No generic gensitivities (model sensitivies only)**
- Unbiased derivative (biased if finite differences are used for w)
- Discontinuous payouts may be handled.

Proxy Scheme Simulation: Sensitivities

Proxy Scheme Sensitivities:

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y^*(\theta)) \mid \mathcal{F}_{T_0}) &\approx \frac{1}{2h}(\mathrm{E}^{\mathbb{Q}}(f(Y^*(\theta+h)) \mid \mathcal{F}_{T_0}) - \mathrm{E}^{\mathbb{Q}}(f(Y^*(\theta-h)) \mid \mathcal{F}_{T_0})) \\
&= \frac{\partial}{\partial\theta}\int_{\mathbb{R}^m} f(y)\cdot\frac{1}{2h}(\phi_{Y^*(\theta+h)}(y) - \phi_{Y^*(\theta-h)}(y))\,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} f(y)\cdot\frac{\frac{1}{2h}(\phi_{Y^*(\theta+h)}(y) - \phi_{Y^*(\theta-h)}(y))}{\phi_{Y^\circ}(y)}\cdot\phi_{Y^\circ}(y)\,\mathrm{d}y \\
&\approx \frac{1}{n}\sum_{i=1}^{n} f(Y^\circ(\omega_i))\cdot\frac{1}{2h}(w(\theta+h,\omega_i) - w(\theta-h,\omega_i))
\end{aligned}$$

Requirements

- Requires no additional information on the model sde $\mathrm{d}X = \dots$
- Requires additional information on the simulation scheme $X^*(T_{i+1})$, $X^\circ(T_{i+1})$
- Requires no additional information on the payout f
- Requires no additional information on the nature of θ ($\Rightarrow$ generic sensitivities)

Properties

- **Generic gensitivities (market sensitivies)**
- Biased derivative (but small shift h possible!)
- Discontinuous payouts may be handled.

Proxy Scheme Simulation: Sensitivities

Proxy Scheme Sensitivities:

$$
\begin{aligned}
\frac{\partial}{\partial\theta}\mathrm{E}^{\mathbb{Q}}(f(Y^{*}(\theta)) \mid \mathcal{F}_{T_0}) &\approx \frac{1}{2h}(\mathrm{E}^{\mathbb{Q}}(f(Y^{*}(\theta+h)) \mid \mathcal{F}_{T_0}) - \mathrm{E}^{\mathbb{Q}}(f(Y^{*}(\theta-h)) \mid \mathcal{F}_{T_0})) \\
&= \frac{\partial}{\partial\theta}\int_{\mathrm{IR}^m} f(y) \cdot \frac{1}{2h}(\phi_{Y^{*}(\theta+h)}(y) - \phi_{Y^{*}(\theta-h)}(y))\,\mathrm{d}y \\
&= \int_{\mathrm{IR}^m} f(y) \cdot \frac{\frac{1}{2h}(\phi_{Y^{*}(\theta+h)}(y) - \phi_{Y^{*}(\theta-h)}(y))}{\phi_{Y^{\circ}}(y)} \cdot \phi_{Y^{\circ}}(y)\,\mathrm{d}y \\
&\approx \frac{1}{n}\sum_{i=1}^{n} f(Y^{\circ}(\omega_i)) \cdot \frac{1}{2h}(w(\theta+h,\omega_i) - w(\theta-h,\omega_i))
\end{aligned}
$$

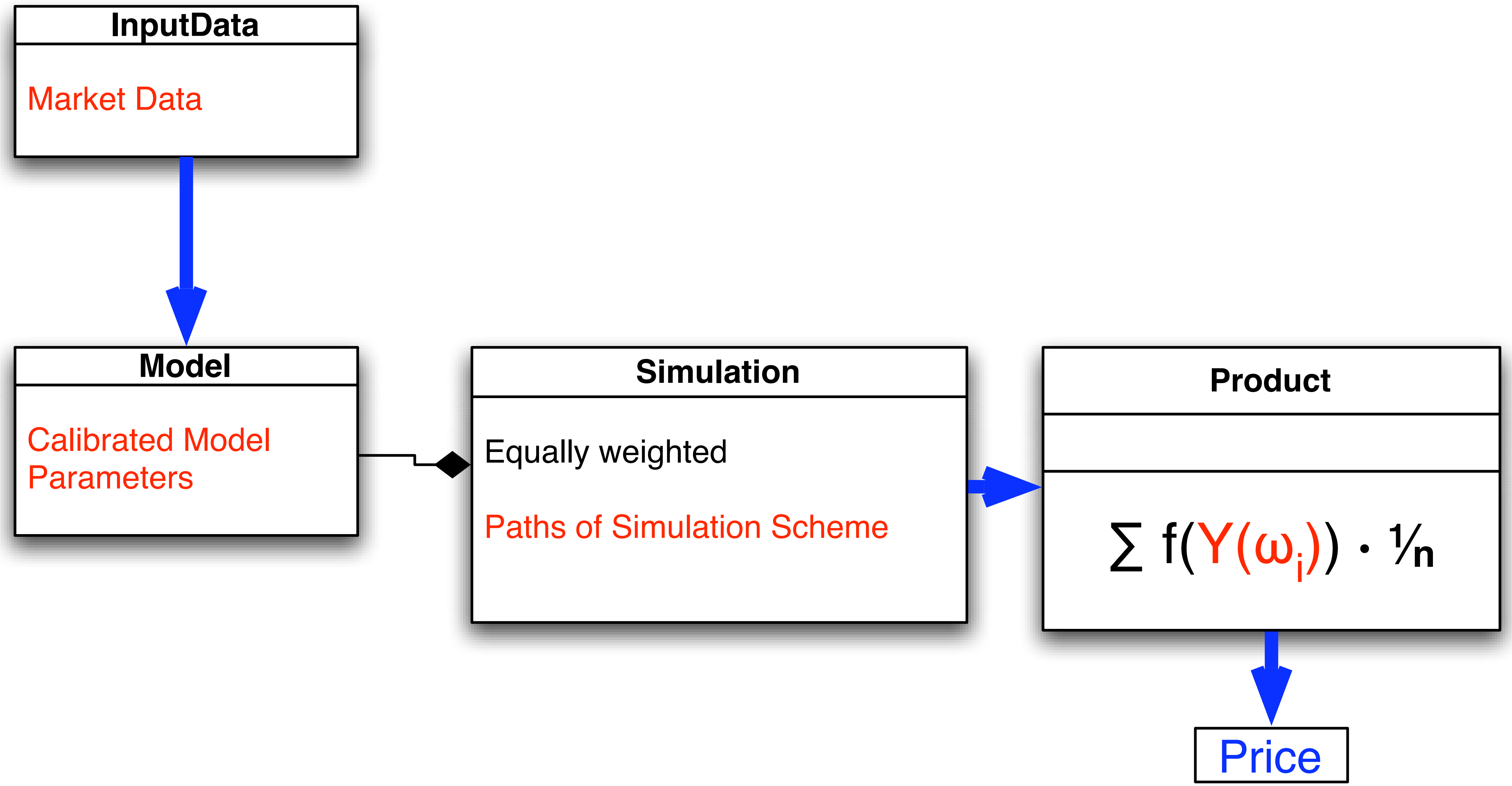
Finite difference applied to the pricing
results in a finite difference approximation of the Likelihood Ratio

thus

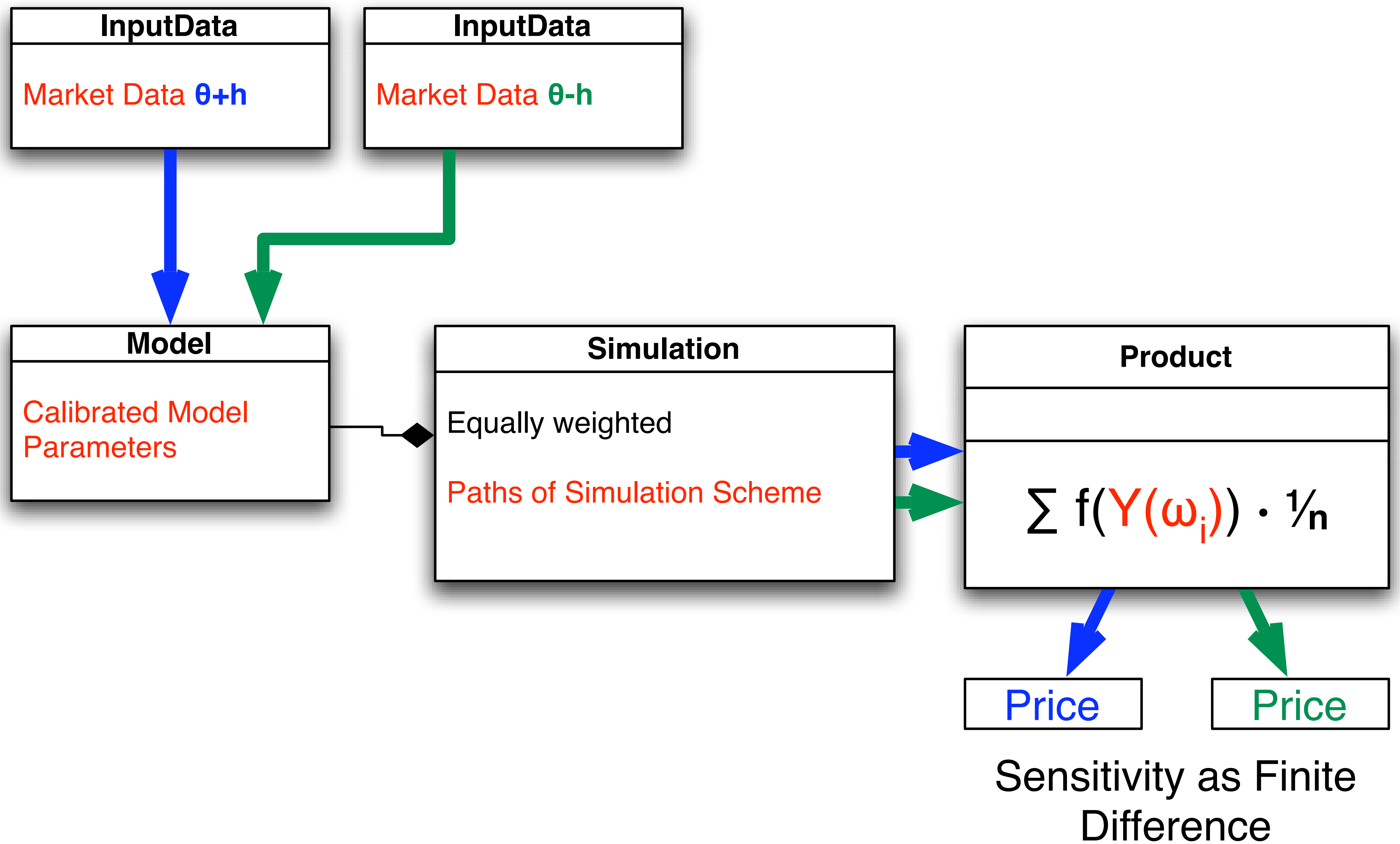
We have all the nice properties of the *Likelihood Ratio*
combined with the genericity of *Finite Differences*

Proxy Simulation Scheme: Implementation

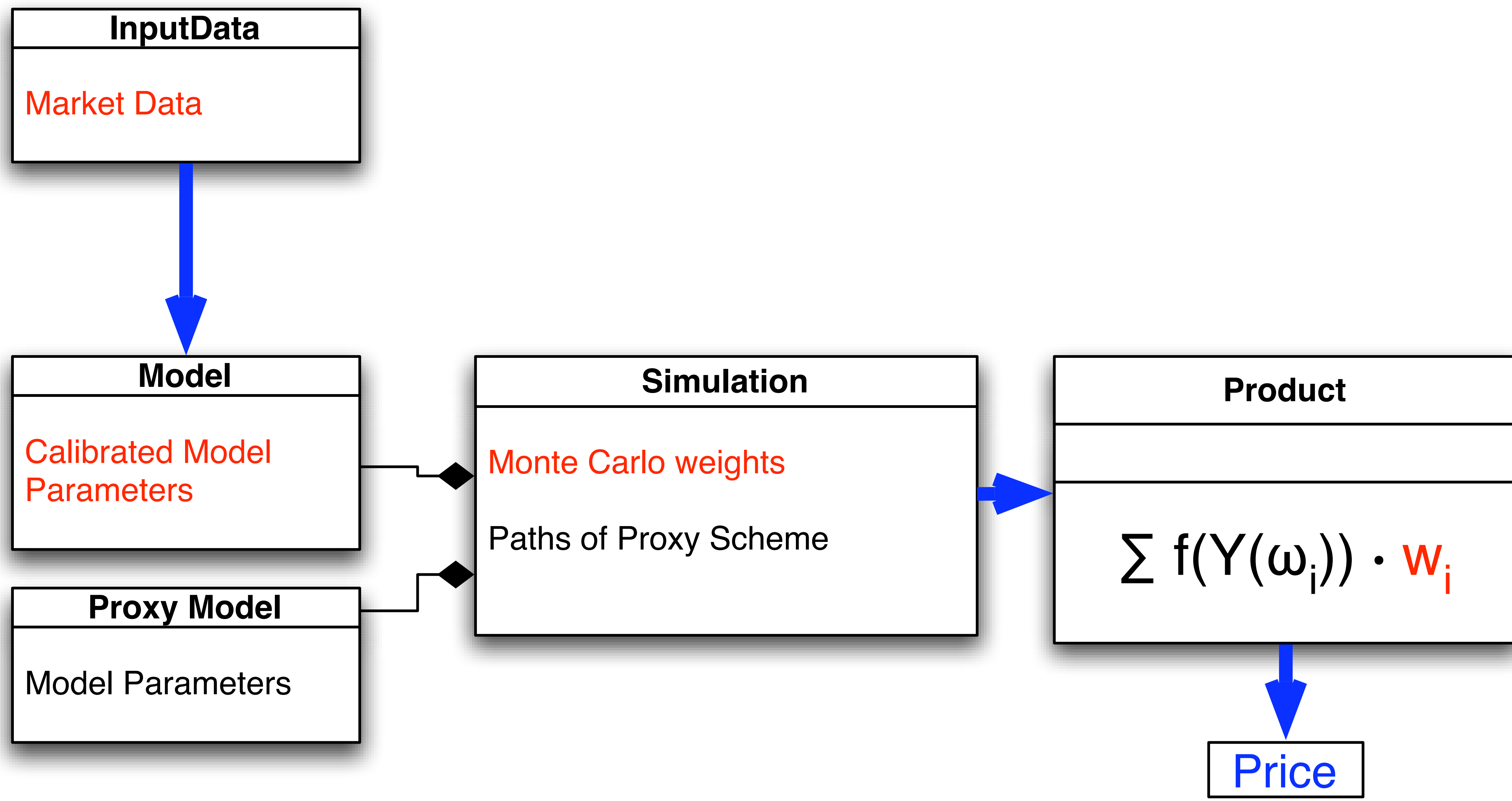
Standard Monte Carlo Simulation: Pricing



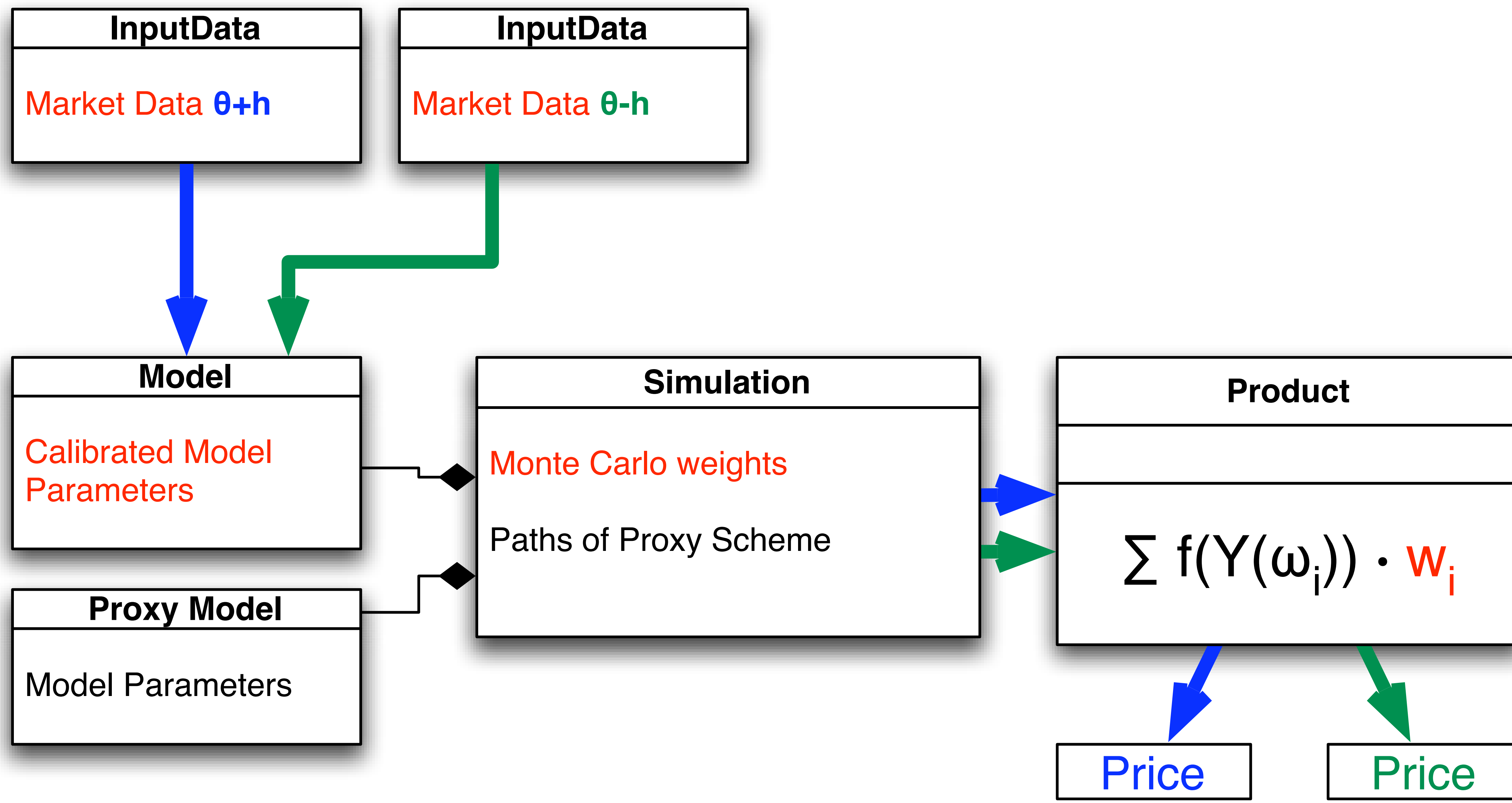
Standard Monte Carlo Simulation: Sensitivities



Proxy Simulation Method: Pricing



Proxy Simulation Method: Sensitivities



LR like Sensitivity as Finite Difference

Proxy Simulation Scheme

A Note on the Monte-Carlo Weights and the Denstities

Proxy Scheme Simulation: Densities / Weak Schemes

Proxy Scheme: Consider *two* numerical schemes

X	$t \mapsto X(t)$	$t \in \mathbb{R}$	model sde
X^*	$T_i \mapsto X^*(T_i)$	$i = 0, 1, 2, \ldots$	time discretization scheme of X → *target scheme*
X°	$T_i \mapsto X^\circ(T_i)$	$i = 0, 1, 2, \ldots$	any other time discrete stochastic process (assumed to be *close* to X^*) → *proxy scheme*

Pricing: Let $Y = (X(T_1), \ldots, X(T_m))$, $Y^* = (X^*(T_1), \ldots, X^*(T_m))$, $Y^\circ = (X^\circ(T_1), \ldots, X^\circ(T_m))$.

$$\mathrm{E}^{\mathbb{Q}}(f(Y(\theta)) \mid \mathcal{F}_{T_0}) \approx \mathrm{E}^{\mathbb{Q}}(f(Y^*(\theta)) \mid \mathcal{F}_{T_0}) = \mathrm{E}^{\mathbb{Q}}(f(Y^\circ) \cdot w(\theta) \mid \mathcal{F}_{T_0})$$

where

$$w(\theta) = \frac{\phi_{Y^*(\theta)}(y)}{\phi_{Y^\circ}(y)} \qquad \text{(calculated numerically).}$$

Note:

- From the scheme X° we need the realizations (to generate the path)
  → Need something explicit (Euler-Scheme, Predictor Corrector, etc.)
- From the scheme X^* we need the transition probability only (weaker requirement)
  → May use complex implicit schemes or expansions of the the transition probability of the (true) model sde.
  Kampen derived a quadratic WKB expansion for the LIBOR Market Model (see appendix)

Summary: Requirements / Implementation

Proxy Scheme Weights:

$$w(T_{i+1}) \mid_{\mathcal{F}_{T_k}} = \prod_{j=k}^{i} \frac{\phi^{K^*}(T_j, K^\circ_j; T_{j+1}, K^\circ_{j+1})}{\phi^{K^\circ}(T_j, K^\circ_j; T_{j+1}, K^\circ_{j+1})}$$

Implementation:

The transition densities ϕ^{K° and ϕ^{K^*} are densities from the numerical schemes K° and K^*. They may be calculated numerically (on the fly together with the (proxy) schemes paths)!

Requirement:

$$\phi^{K^\circ}(T_i, K^\circ_i; T_{i+1}, K^\circ_{i+1}) \;=\; 0 \;\Longrightarrow\; \phi^{K^*}(T_i, K^\circ_i; T_{i+1}, K^\circ_{i+1}) \;=\; 0$$

This requirement corresponds to the non-degeneracy condition imposed on the diffusion matrix in the continuouse case (e.g. Malliavin Calculus).

However: Here, this requirement may be achieved even for a degenerate diffusion matrix, e.g. by a non-linear drift.

Moreover:

Since we are free to choose the proxy sheme, it may choosen such that the condition holds.

Summary

Summary: Properties / Achivements

Requirements:

- Requires no additional information on the model sde $\mathrm{d}X = \ldots$
- Requires additional information on the simulation scheme $X^{*}(T_{i+1}),\ X^{\circ}(T_{i+1})$
- Requires no additional information on the payout f
- Requires no additional information on the nature of θ ($\Rightarrow$ generic sensitivities)
- Stable for small shifts h
- Discontinuous payouts may be handled.

Achievements:

- **Stable Generic Sensitivities:** Finite Differences result in numerical Likelihood Ratios
- **Weak Schemes:** Allows to correct for an improper transition density.

Example:
LIBOR Market Model

Example: Proxy Scheme Simulation for a LIBOR Market Model

LIBOR Market Model:

$$\mathrm{d}L_i = L_i\,\mu_i^L\,\mathrm{d}t \;+\; L_i\,\sigma_i\,\mathrm{d}W_i, \quad i = 1,\ldots n, \quad \text{with} \quad \mu_i^L = \sum_{i<j\leq n} \frac{L_j \delta_j}{1+L_j\delta_j}\sigma_i\sigma_j\rho_{i,j}, \qquad \mathrm{d}W = \Sigma\cdot\Gamma\cdot\mathrm{d}U,$$

where $\mathrm{d}W = (\mathrm{d}W_1,\ldots,\mathrm{d}W_n)$, $\quad \mathrm{d}W_i\,\mathrm{d}W_j = \rho_{i,j}\,\mathrm{d}t$, $\quad \Sigma = \mathrm{diag}(\sigma_1,\ldots,\sigma_n)$, $\quad \Gamma\,\Gamma^\mathsf{T} = (\rho_{i,j})$.

- Log-normal model (common extensions: local vol., stoch. vol., jump)
- Non-linear drift
- High dimensional (no low dimensional Markovian state variable)
- Driving factors may be low dimensional (parsimonious model) $\rightarrow$ Γ is an $n\times m$ matrix.

LIBOR Market Model & Numerical Schemes in Log-Coordinates:

model sde: $\mathrm{d}K \;=\; \mu^K\,\mathrm{d}t \;+\; \Sigma\cdot\Gamma\cdot\mathrm{d}U \qquad K := \log(L), \quad \mu^K := \mu^L - \frac{1}{2}\Sigma^2$

proxy scheme: $K^\circ(T_{i+1}) \;=\; K^\circ(T_i) \;+\; \mu^{K^\circ}(T_i)\Delta T_i \;+\; \Sigma^\circ(T_i)\cdot\Gamma^\circ(T_i)\cdot\Delta U(T_i)$

target scheme: $K^*(T_{i+1}) \;=\; K^*(T_i) \;+\; \mu^{K^*}(T_i)\Delta T_i \;+\; \Sigma(T_i)\cdot\Gamma(T_i)\cdot\Delta U(T_i)$

Example: Proxy Scheme Simulation for a LIBOR Market Model

LIBOR Market Model & Numerical Schemes in Log-Coordinates:

model sde: $\mathrm{d}K = \mu^K \,\mathrm{d}t + \Sigma \cdot \Gamma \cdot \mathrm{d}U \qquad K := \log(L), \quad \mu^K := \mu^L - \frac{1}{2}\Sigma^2$

proxy scheme: $K^\circ(T_{i+1}) = K^\circ(T_i) + \mu^{K^\circ}(T_i)\Delta T_i + \Sigma^\circ(T_i) \cdot \Gamma^\circ(T_i) \cdot \Delta U(T_i)$ $\longleftarrow$ sample path

target scheme: $K^*(T_{i+1}) = K^*(T_i) + \mu^{K^*}(T_i)\Delta T_i + \Sigma(T_i) \cdot \Gamma(T_i) \cdot \Delta U(T_i)$

Tansition Probabilites $T_i \to T_{i+1}$:
Assume for simplicity that $\mu^{K^*}(T_i)$ depends on $K^*(T_i)$, $K^*(T_{i+1})$ only (and same for $^\circ$)
($\rightarrow$ true for, e.g. Euler Scheme, Predictor Corrector), then

$$\phi^{K^\circ}(T_i, K_i^\circ; T_{i+1}, K_{i+1}^\circ) = \frac{1}{(2\Pi\Delta T_i)^{n/2}} \exp\Big(-\frac{1}{2\Delta T_i} (\Lambda^{\circ -1/2} F^{\circ \mathrm{T}} \Sigma^{\circ -1}(K_{i+1}^\circ - K_i^\circ - \mu^{K^\circ}(T_i)\Delta T_i))^2 \Big)$$

$$\phi^{K^*}(T_i, K_i^*; T_{i+1}, K_{i+1}^*) = \frac{1}{(2\Pi\Delta T_i)^{n/2}} \exp\Big(-\frac{1}{2\Delta T_i} (\Lambda^{-1/2} F^{\mathrm{T}} \Sigma^{-1}(K_{i+1}^* - K_i^* - \mu^{K^*}(T_i)\Delta T_i))^2 \Big)$$

Proxy Scheme Weights:

$$w(T_{i+1}) \,|_{\mathcal{F}_{T_k}} = \prod_{j=k}^{i} \frac{\phi^{K^*}(T_j, K_j^\circ; T_{j+1}, K_{j+1}^\circ)}{\phi^{K^\circ}(T_j, K_j^\circ; T_{j+1}, K_{j+1}^\circ)}$$ $\longleftarrow$ monte carlo weights

Note: We used the factor decomposition (PCA) $\Gamma = F \cdot \sqrt{\Lambda}$ where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ are the non-zero Eigenvalues of $\Gamma \cdot \Gamma^{\mathrm{T}}$.

A change of market data / calibration enters into transition probabilities only.

Examples and Numerical Results

Numerical Results

Proxy Scheme: Consider *three* stochastic processes

X	$t \mapsto X(t)$	$t \in \mathbb{R}$	model sde
X^*	$T_i \mapsto X^*(T_i)$	$i = 0, 1, 2, \ldots$	time discretization scheme of X → *target scheme*
X°	$T_i \mapsto X^\circ(T_i)$	$i = 0, 1, 2, \ldots$	any other time discrete stochastic process (assumed to be *close* to X^*) → *proxy scheme*

Test Case:

X LIBOR Market Model

X^* Target Scheme: Some standard discretization of LMM.

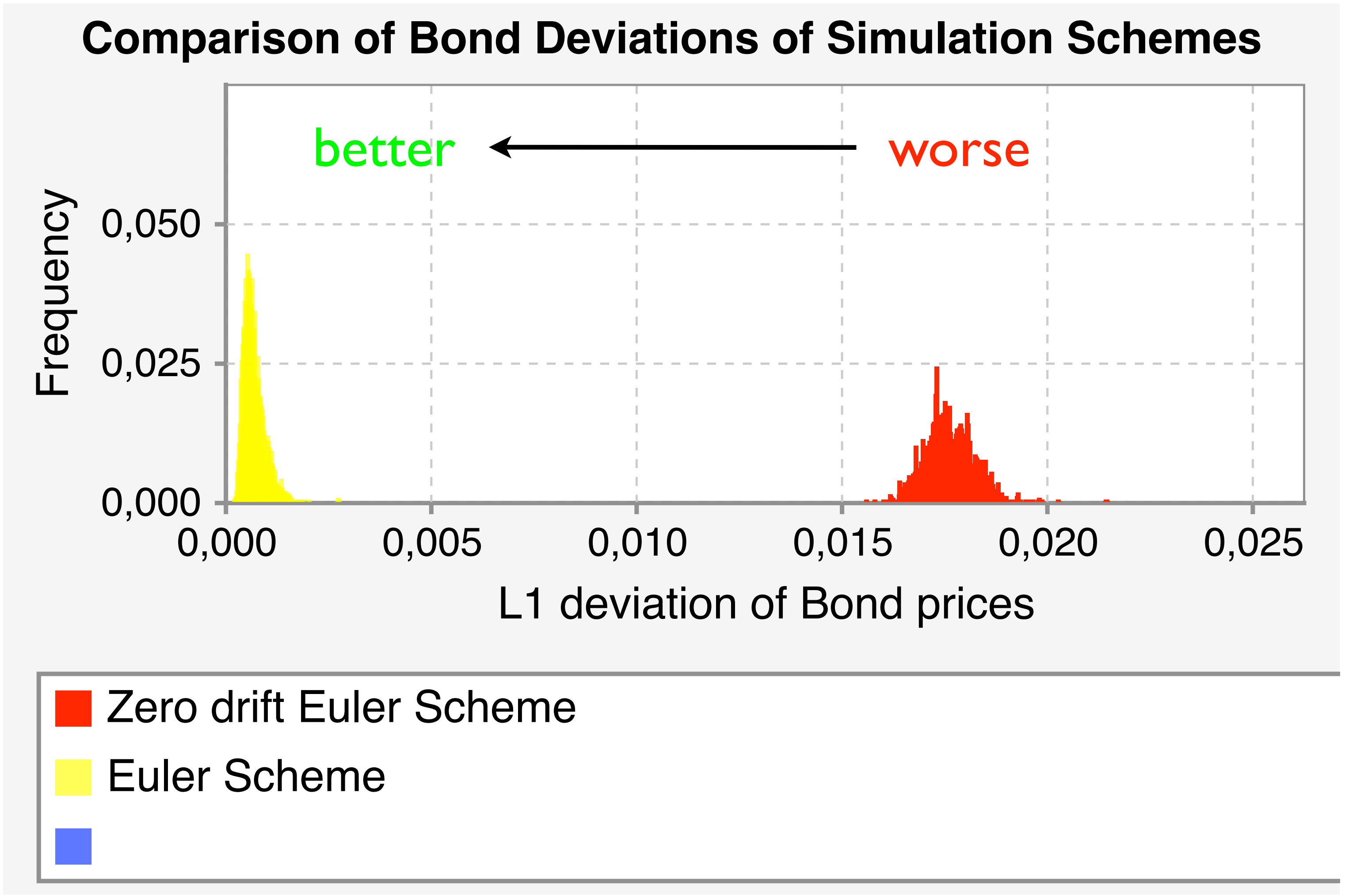
X° Proxy Scheme: Log-normal scheme without drift (LMM drift zero) (extrem test case).

Check for:

- Bond prices (⇔ can we correct for the drift)
- Sensitivities of Trigger Products (Digitals, Auto Caps)

Example 1:
Correcting the Drift

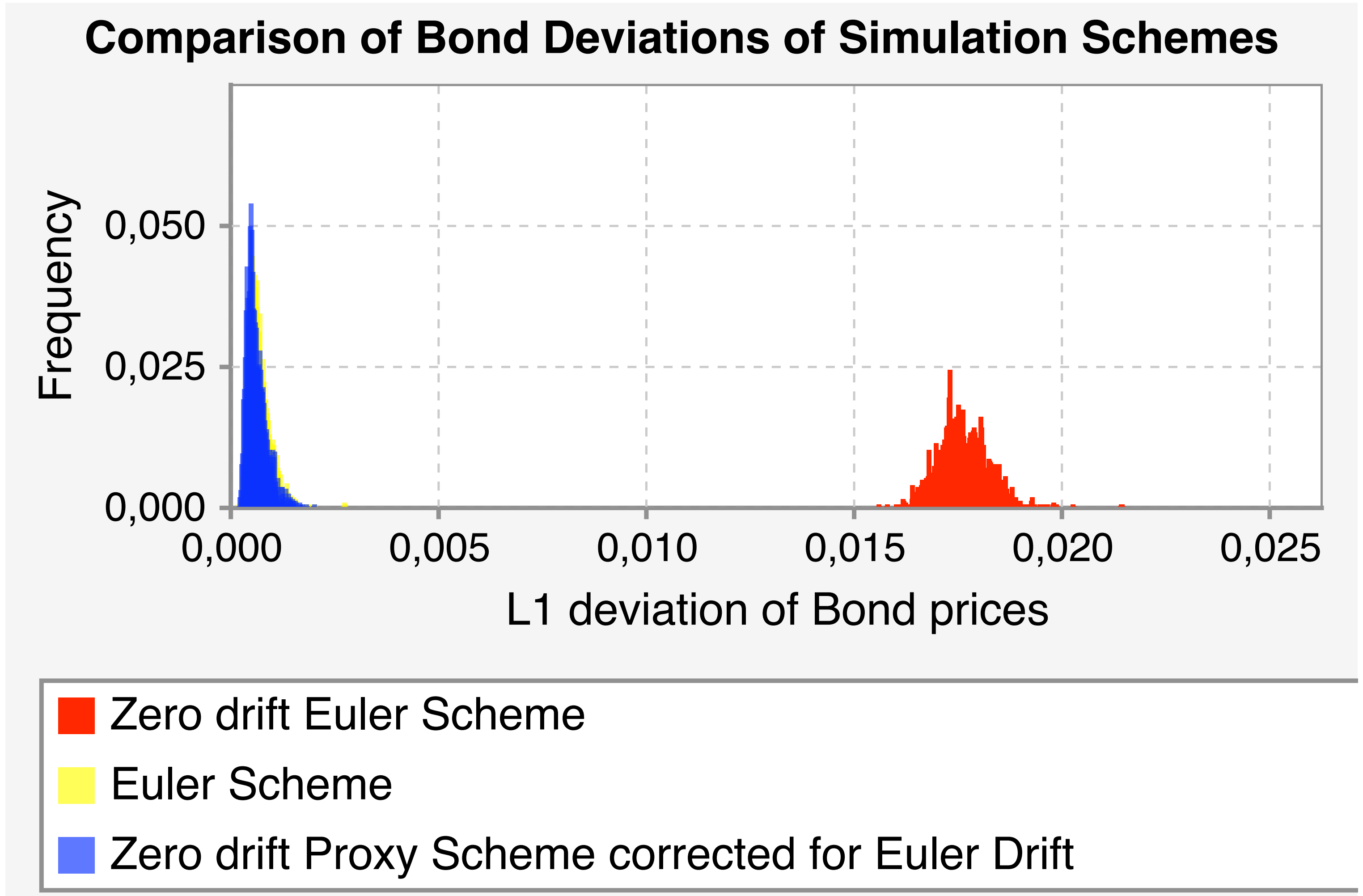
Numerical Results: Monte Carlo Bond Price Distributions



- Shown: Absolute Bond price Monte Carlo error distribution for Euler Scheme with drift zero (red) and Euler Scheme with Euler drift (yellow): Neglecting drift results in large Bond price errors and even higher Monte Carlo variance (since here drift would generate mean reversion).

- Next: Use zero-drift Euler Scheme as proxy scheme and correct drift towards Euler Scheme with drift (target scheme).

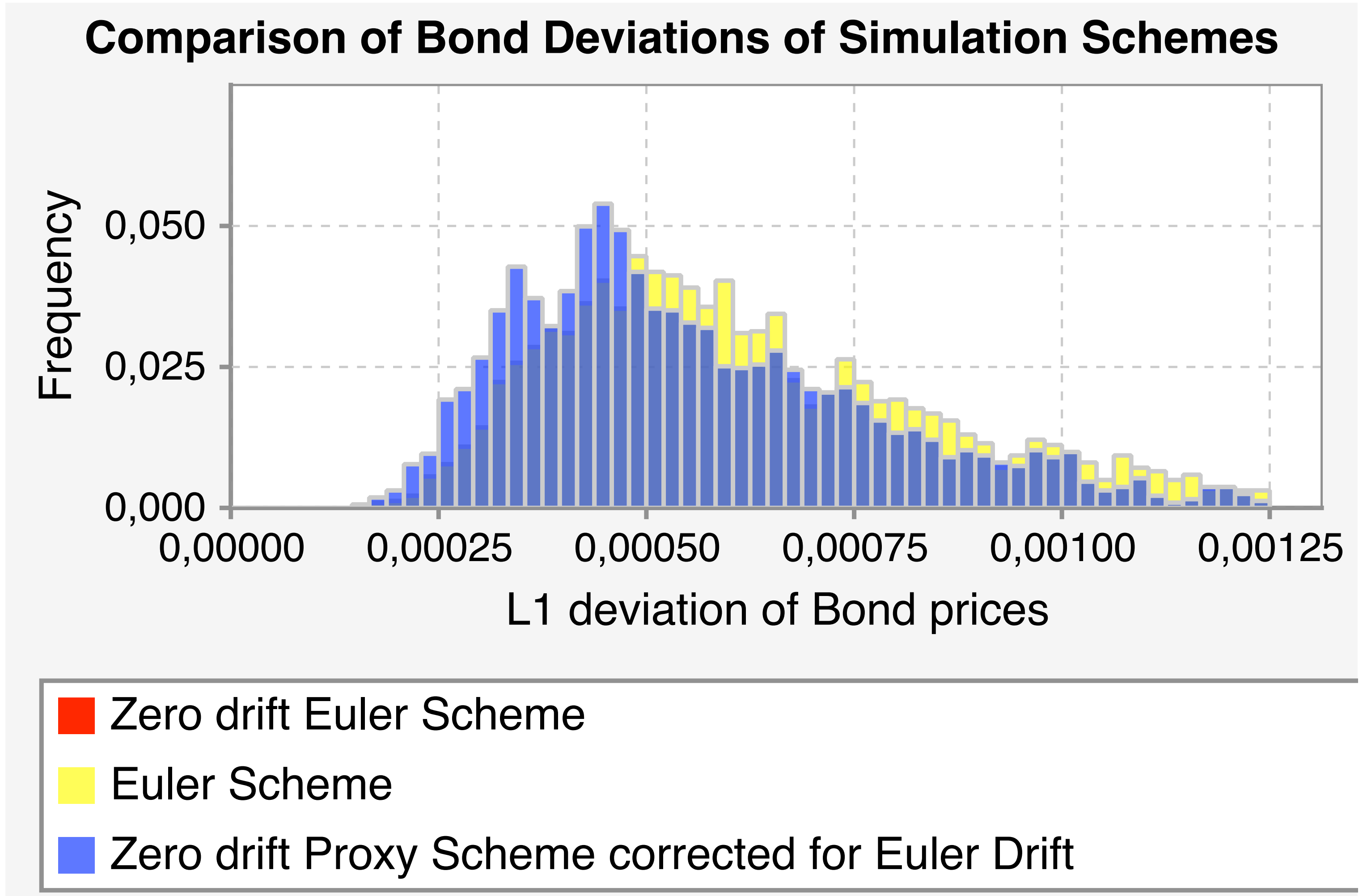
Numerical Results: Monte Carlo Bond Price Distributions



- Shown: Use zero-drift Euler Scheme as proxy scheme (red) and correct drift towards Euler Scheme with drift (target scheme, blue)

- Next: Take a closer look. Compare proxy scheme simulation with direct simulation

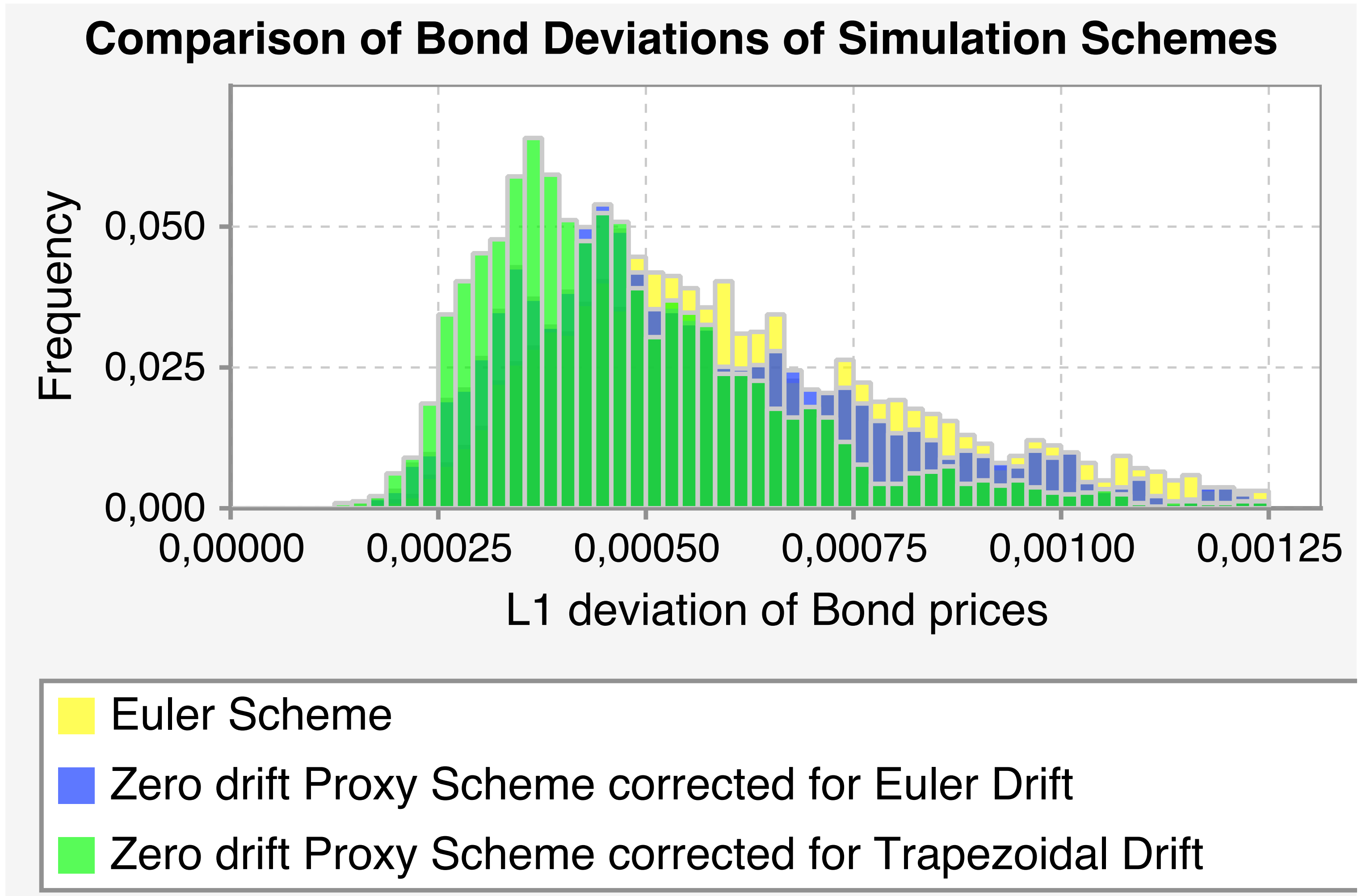
Numerical Results: Monte Carlo Bond Price Distributions



- Shown: Monte Carlo Error of Bond Prices for Proxy-Scheme Method (using zero-drift Euler Scheme as proxy scheme) (blue) and direct simulation of target scheme (yellow)

- Next: Refine target scheme by more accurate transition probabilities

Numerical Results: Monte Carlo Bond Price Distributions



- Shown: Direct Euler Scheme simulation (yellow), Proxy Sheme simulation with Euler Scheme as target scheme (blue), Proxy Scheme simulation with transition probablities derived from trapezoidal integration rule for the drift (green).

Example 2:
Importance Sampling

Proxy Simulation Scheme: Importance Sampling

Importance Samling: The key idea of importance sampling is to generate the paths according to their importance to the application, not according to their probability law, and adjust towards their probability by a suitable Monte-Carlo weight.

Using a proxy simulation scheme, the paths are generated according to the proxy scheme while a Monte-Carlo weight adjusts their probability towards the target scheme.

Choosing the proxy scheme such thats it creates paths according to their importance to the application thus is a form of importance sampling.

Advantage: Specifying a process is easy, the Monte-Carlo weights are calculated automatically by the proxy scheme framework.

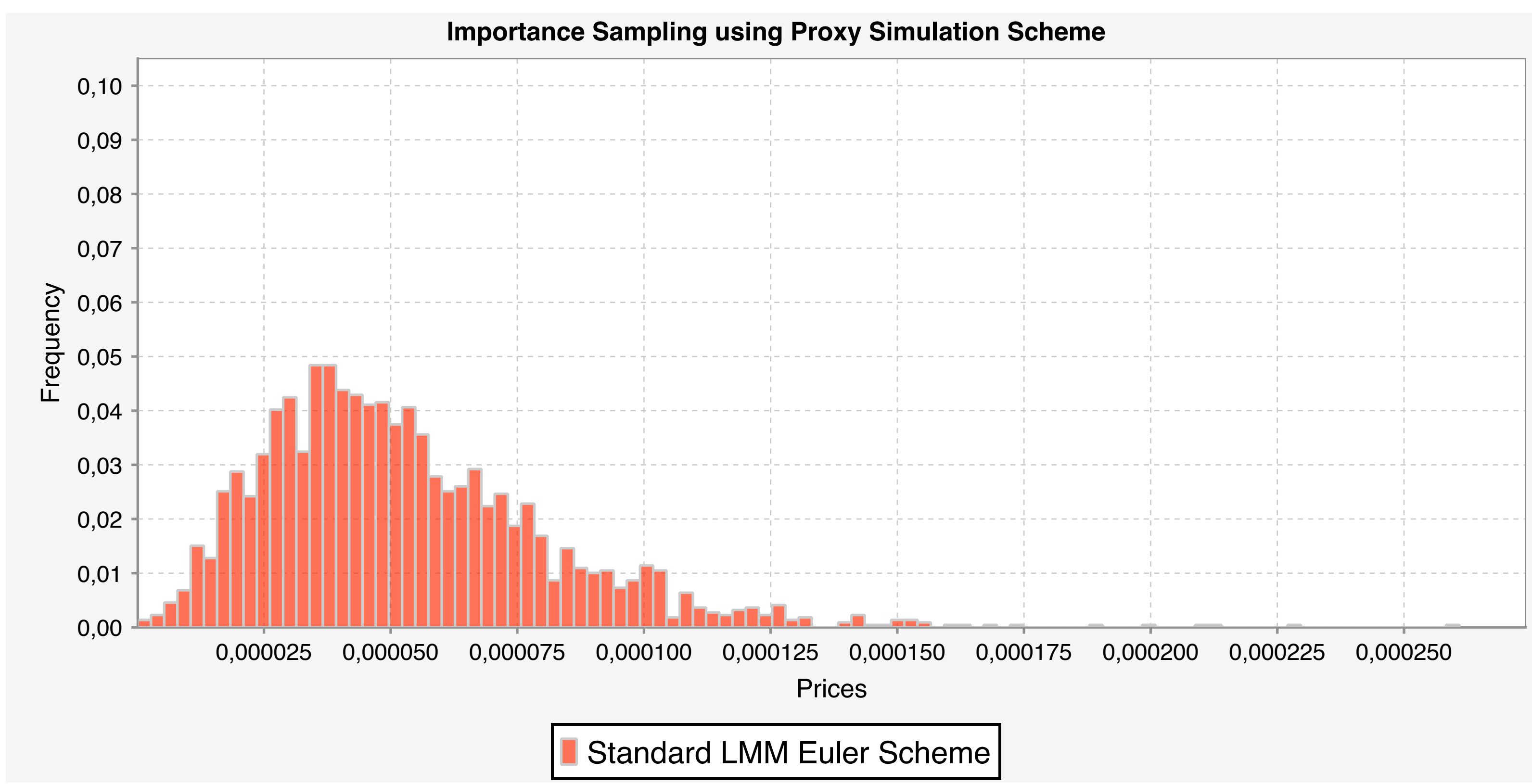
Proxy Simulation Scheme: Importance Sampling

Example: Pricing of a far out of the money caplet under a LIBOR Market Model:

Log Euler Scheme: $\mathrm{d}\log(L_i)(t_{j+1}) = \log(L_i)(t_{j+1}) + \mu_i(t)\mathrm{d}t + \sigma_i \mathrm{d}W_i$ $\quad L_i(0) = 0.1$

OTM caplet: $\max(L(T_i; T_i) - K, 0),$ $\quad K = 0.3$

The drit of the LIBOR Market Model is determined by the pricing measure. However, in our application we would prefer that the mean of $L_i(T_i)$ is close to the option strike $K = 0.3$ rather than $L_i(0) + \int_0^{T_i} \mu_i(t)\mathrm{d}t$. To achieve this, simply use a proxy scheme with artificial drift:



Proxy Simulation Scheme: Importance Sampling

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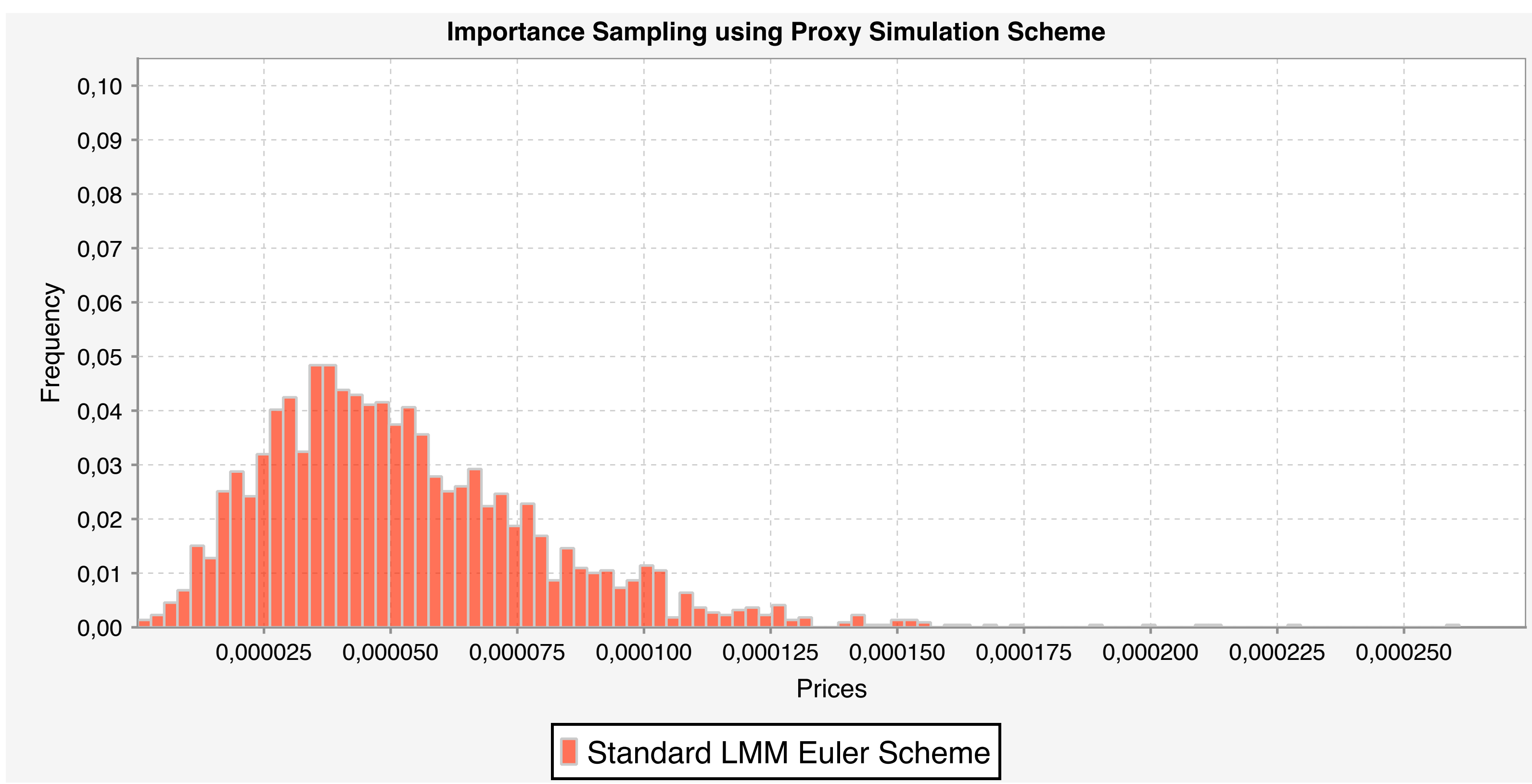
Proxy Scheme:	$\mathrm{d}\log(L_i)(t_{j+1}) = \log(L_i)(t_{j+1}) + \frac{\log(K) - \log(L_i(0))}{T}\mathrm{d}t + \sigma_i \mathrm{d}W_i$	$L_i(0) = 0.1$
Target Scheme:	$\mathrm{d}\log(L_i)(t_{j+1}) = \log(L_i)(t_{j+1}) + \mu_i(t)\mathrm{d}t + \sigma_i \mathrm{d}W_i$	$L_i(0) = 0.1$



Proxy Simulation Scheme: Importance Sampling

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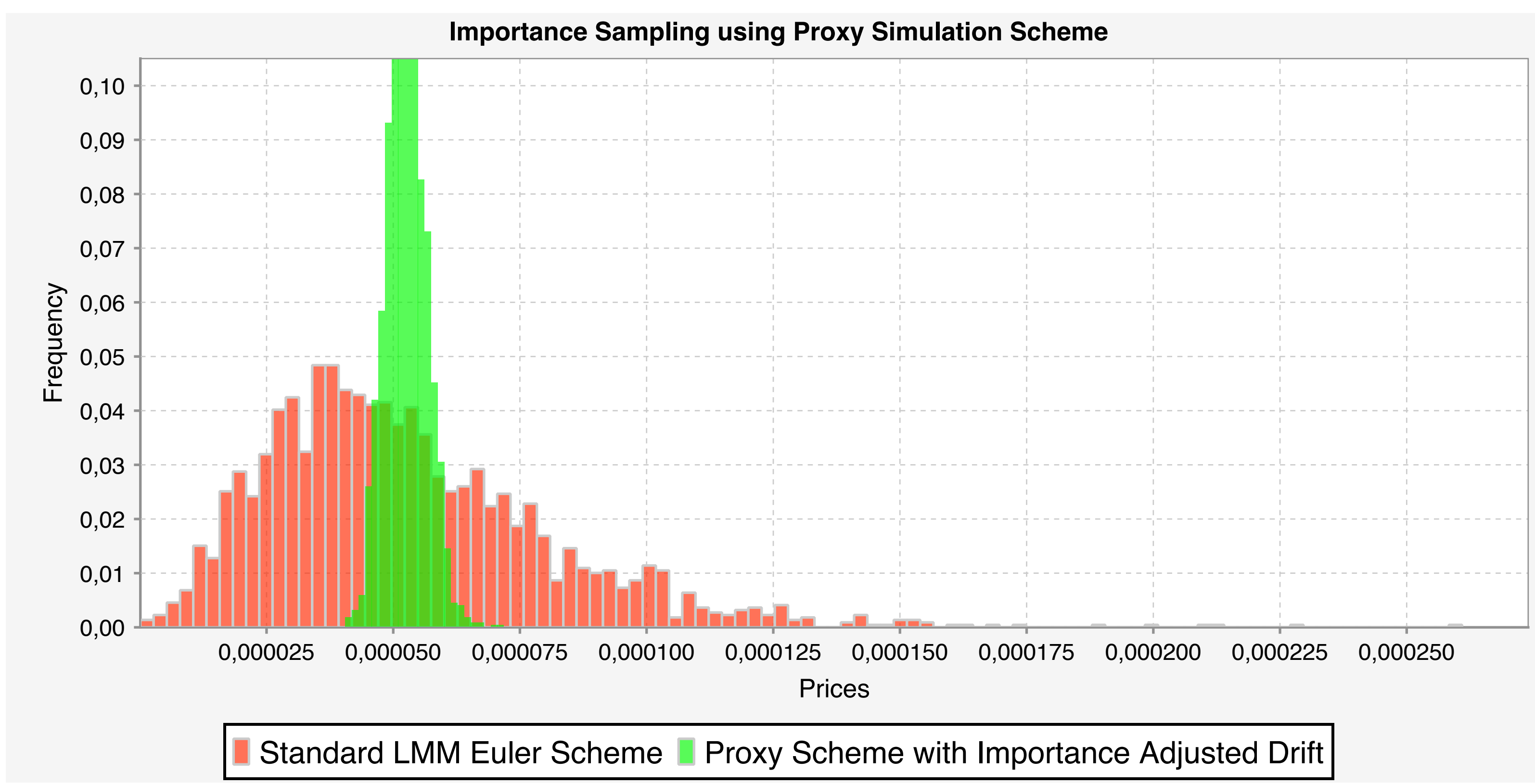
OTM caplet:	$\max(L(T_i;T_i) - K, 0),$	$K = 0.3$

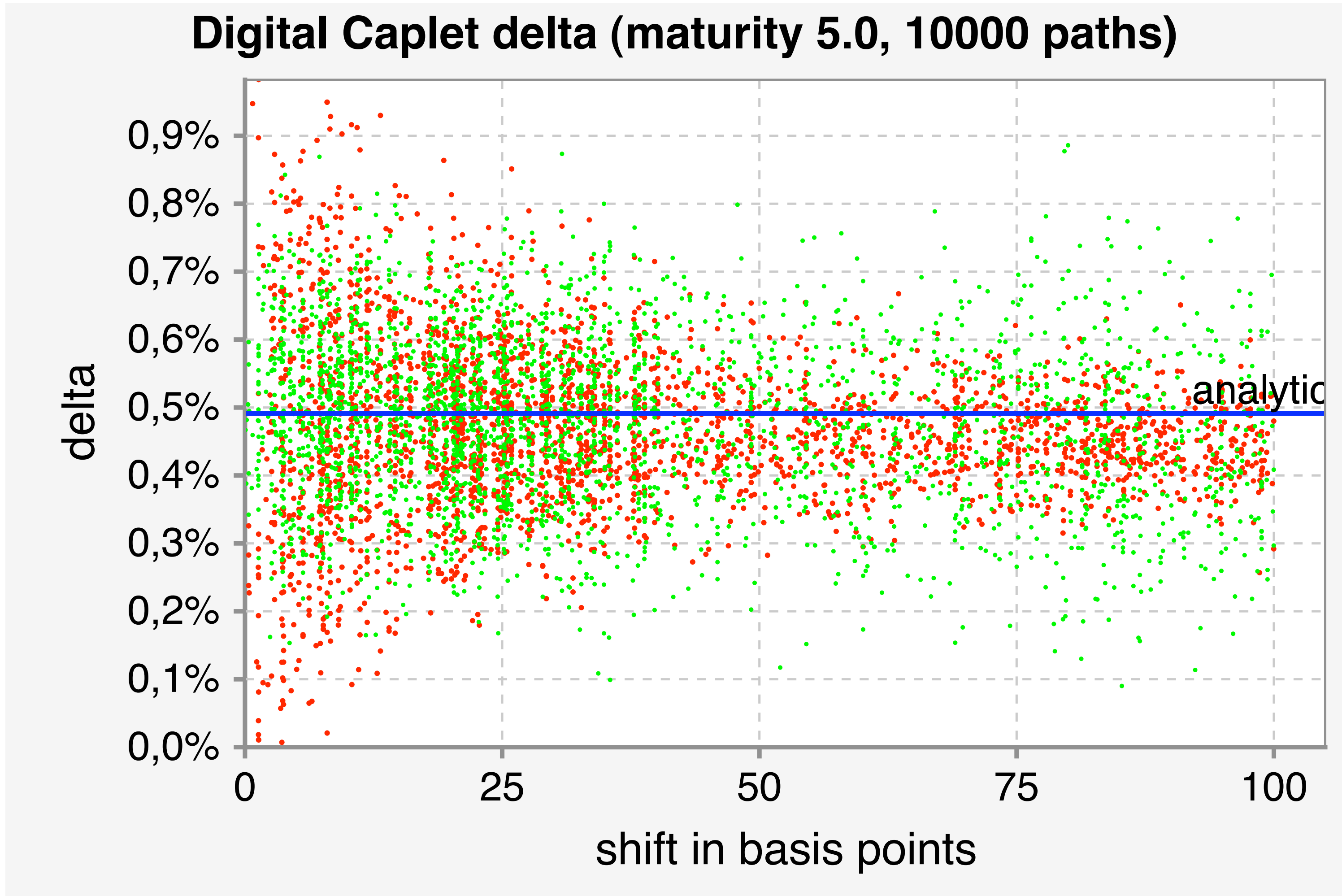
The drit of the LIBOR Market Model is determined by the pricing measure. However, in our application we would prefer that the mean of $L_i(T_i)$ is close to the option strike $K = 0.3$ rather than $L_i(0) + \int_0^{T_i} \mu_i(t)\mathrm{d}t$. To achieve this, simply use a proxy scheme with artificial drift:

Proxy Scheme:	$\mathrm{d}\log(L_i)(t_{j+1}) = \log(L_i)(t_{j+1}) + \frac{\log(K) - \log(L_i(0))}{T}\mathrm{d}t + \sigma_i \mathrm{d}W_i$	$L_i(0) = 0.1$
Target Scheme:	$\mathrm{d}\log(L_i)(t_{j+1}) = \log(L_i)(t_{j+1}) + \mu_i(t)\mathrm{d}t + \sigma_i \mathrm{d}W_i$	$L_i(0) = 0.1$



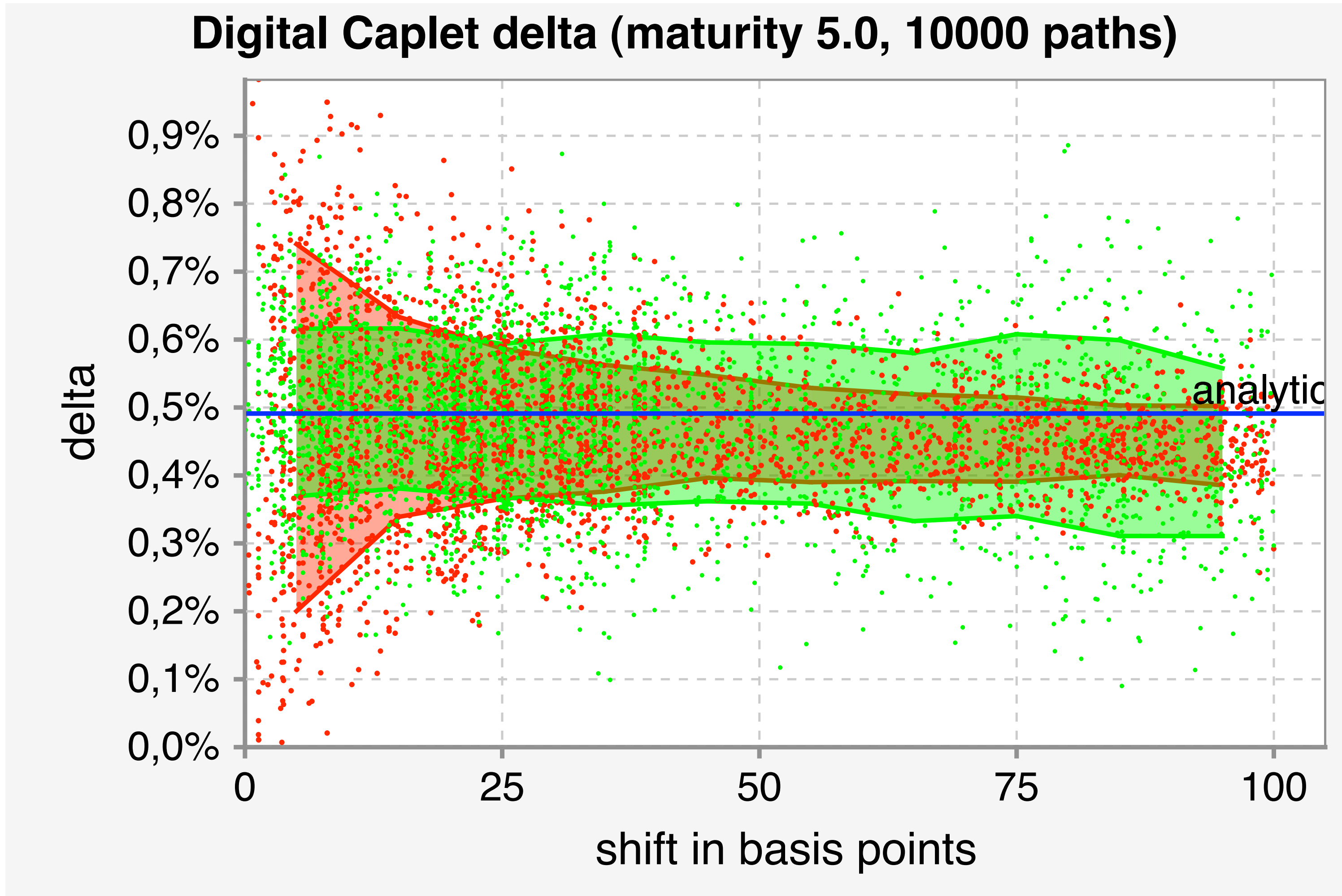
Example 3:
Robust Generic Sensitivities

Numerical Results: Monte Carlo Sensitivities



- Proxy Scheme Sensitivity shows an increase of variance for large shift (well known effect for Likelihood Ratio / Malliavin Calculus)
- Proxy Scheme Sensitivity remains stable for small shifts

Numerical Results: Monte Carlo Sensitivities



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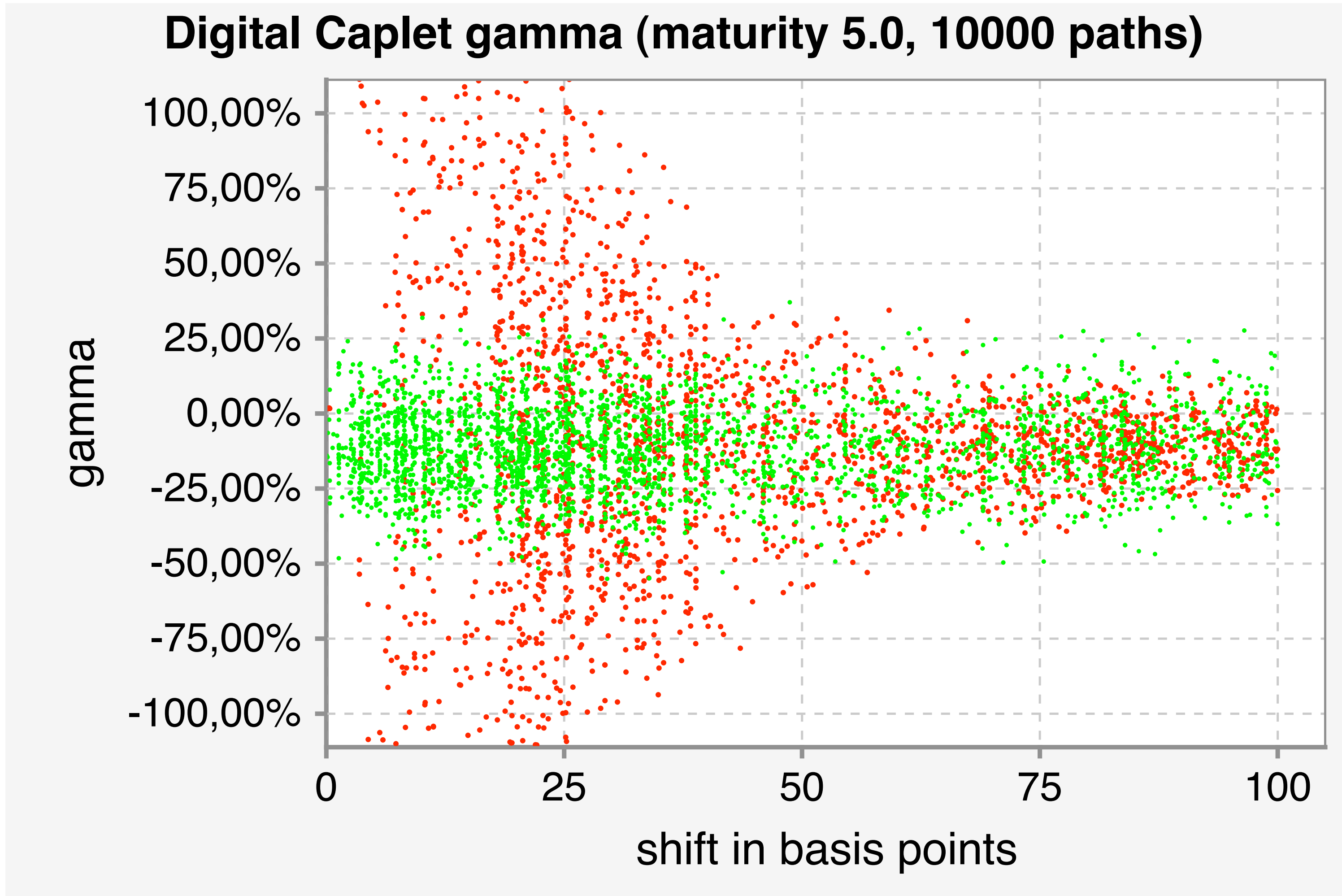
Numerical Results: Monte Carlo Sensitivities



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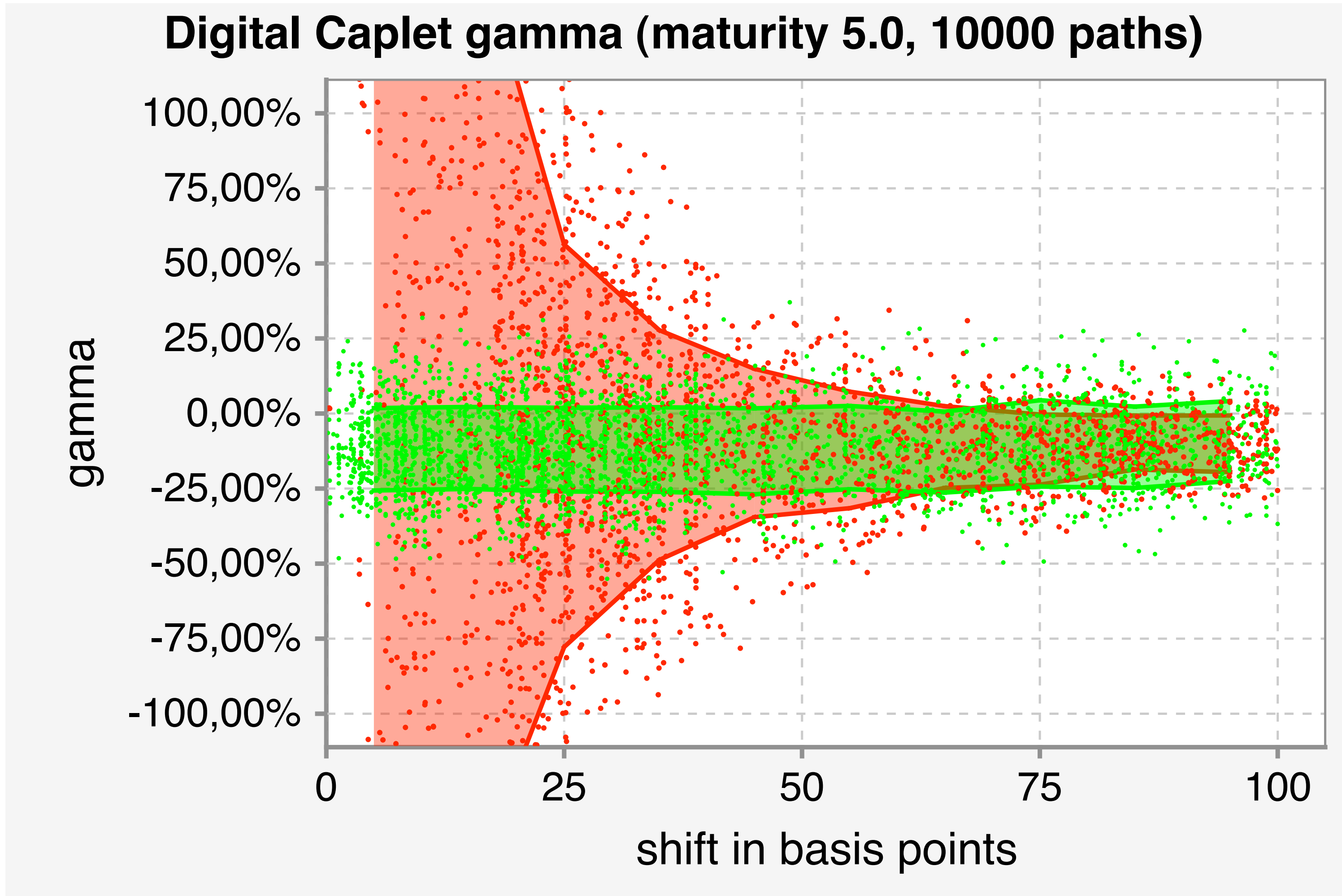
Numerical Results: Monte Carlo Sensitivities



- Proxy Scheme Sensitivity shows an increase of variance for large shift (well known effect for Likelihood Ratio / Malliavin Calculus)
- Proxy Scheme Sensitivity remains stable for small shifts

Drawbacks

Drawback:

Proxy Scheme
is required to be measure equivalent to Target Scheme

Drawbacks: Note on the non-degeneracy condition (1/2)

A note on the requirement

$$\forall y : \phi^{Y^\circ}(y) \;=\; 0 \;\Rightarrow\; \phi^{Y^*}(y) \;=\; 0 \qquad (*)$$

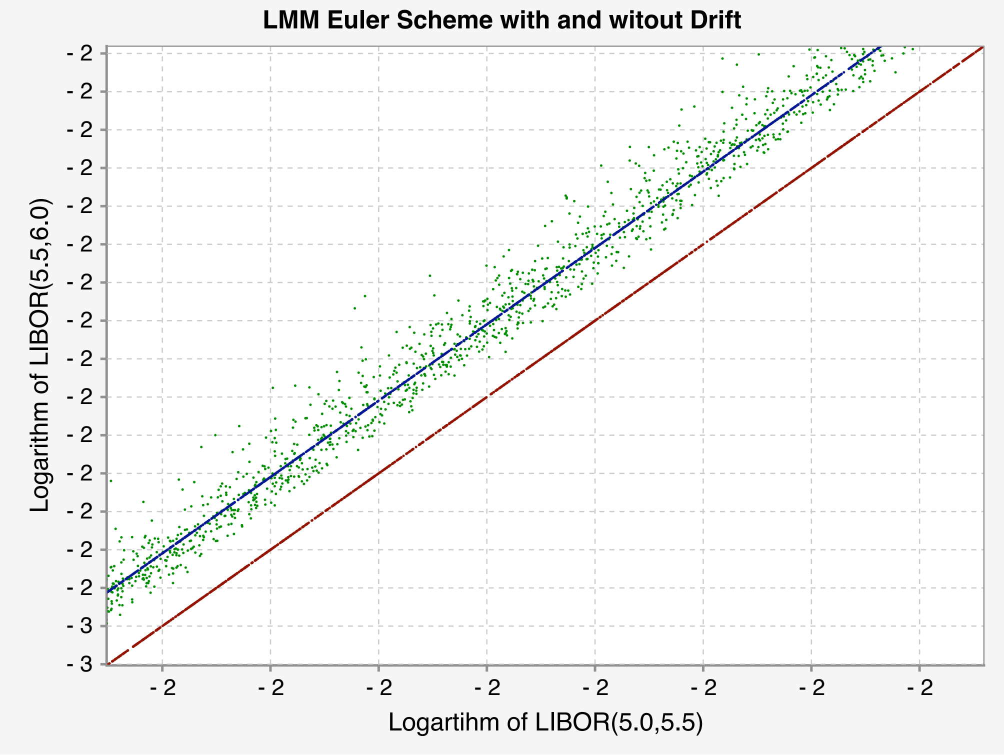
The condition ensures that calculating an expectation on (weighted) paths Y° may be equivalent to calculation expectation on paths Y^*. No Y^*-path is missing.

Question: Is it possible to fulfill this condition in general? What happens if the condition is violated?

Observation 1: While for Malliavin Calculus one would expect some non-degeneracy condition imposed on the diffusion matrix. Here, condition (*) *is much weaker*. Since we may choose the (time-discrete) simulation scheme we may make (*) hold. Either add artificial diffusion or use multiple euler steps:

Example:

Consider a model on two state variable (here an LMM) with a degenerate (rank 1) diffusion matrix (red) and a stochastic drift term (like in LMM). Then a single Euler step will span a line (blue). Using this as a proxy scheme will not allow drift corrections outside that 1-dim hypersurface. However, two subsequent Euler steps of half the size, generate diffusion perpendicular to the 1-dim hypersurface (green). See [F06].



Drawbacks: Note on the non-degeneracy condition (2/2)

A note on the requirement

$$\forall y : \phi^{Y^\circ}(y) \;=\; 0 \;\Rightarrow\; \phi^{Y^*}(y) \;=\; 0 \qquad (*)$$

Observation 2: Since we use the proxy scheme to generate the paths $Y^\circ(\omega)$ we trivially have

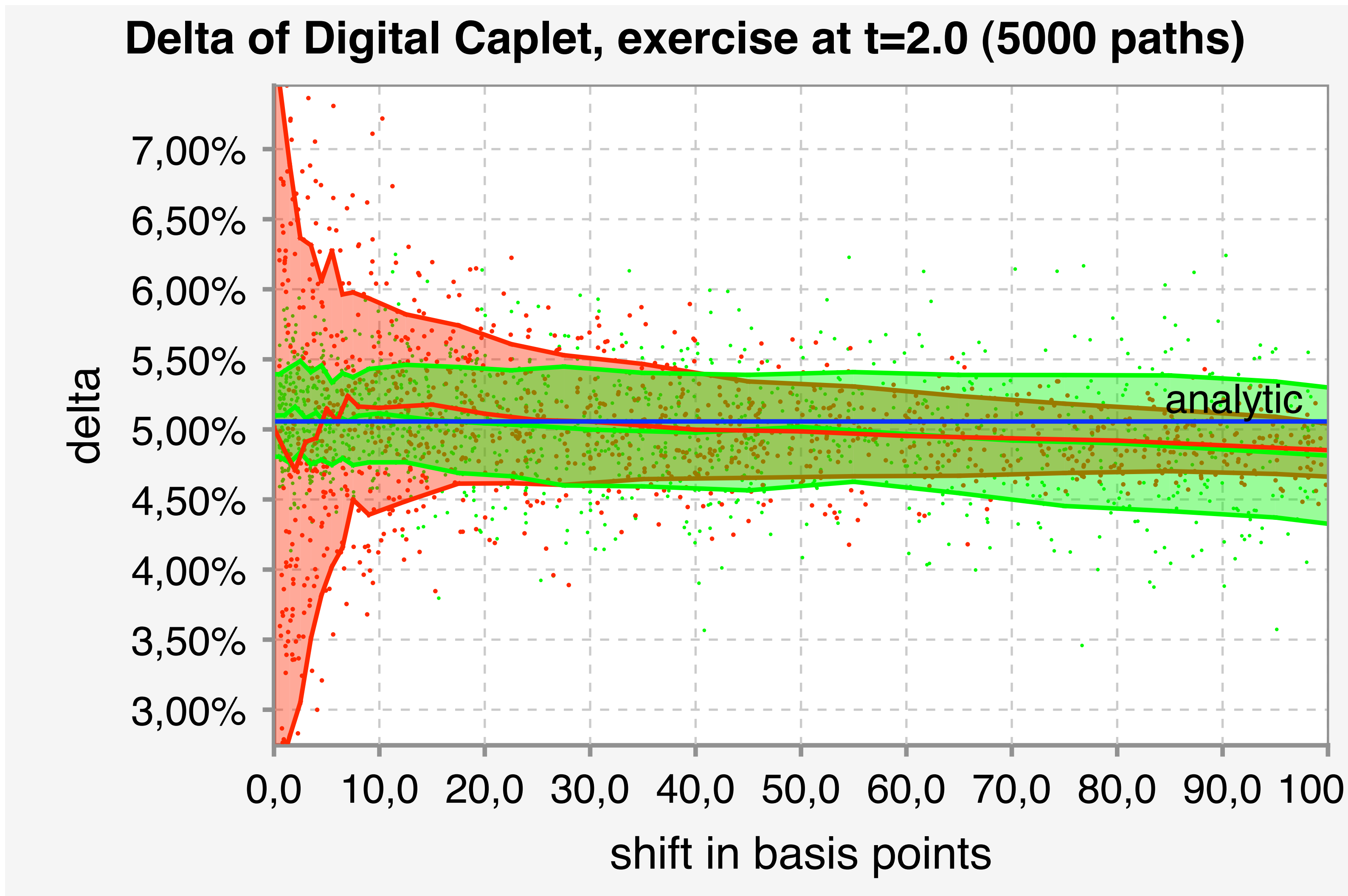
$$\phi^{Y^\circ}(Y^\circ(\omega)) \;\neq\; 0 \qquad \text{on all paths } \omega \text{ generated.}$$

Thus the implementation will never suffer from a division by zero error. So how about neglecting condition (*).

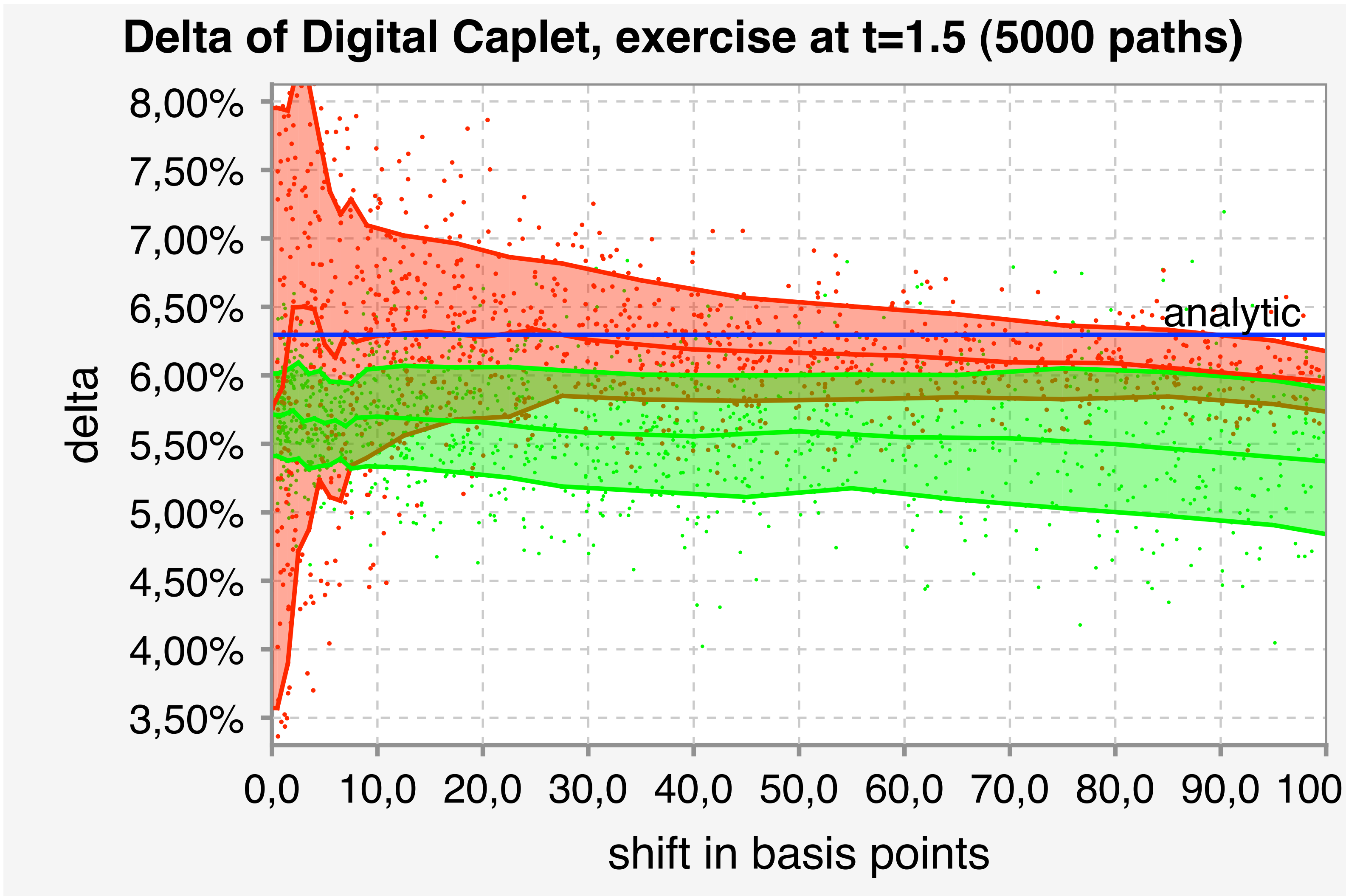
Observation 3: If the requirement (*) does not hold, then the expectation $\mathrm{E}^{\mathbb{Q}}\Big(f(Y^\circ) \cdot \frac{\phi_{Y^*}(Y^\circ)}{\phi_{Y^\circ}(Y^\circ)} \mid \mathcal{F}_{T_0}\Big)$ will leave out some mass. If the two schemes are close, this missed mass is small. In addition one may numerically correct for the missed mass.

Note: If we are in the setup of sensitivities and ϕ^{Y^*} is a scenario perturbation of ϕ^{Y°, then a violation of (*) means that the scenario is impossible under the original model. Either the relevance of the scenario or the explanatory power of the model should be put into question.

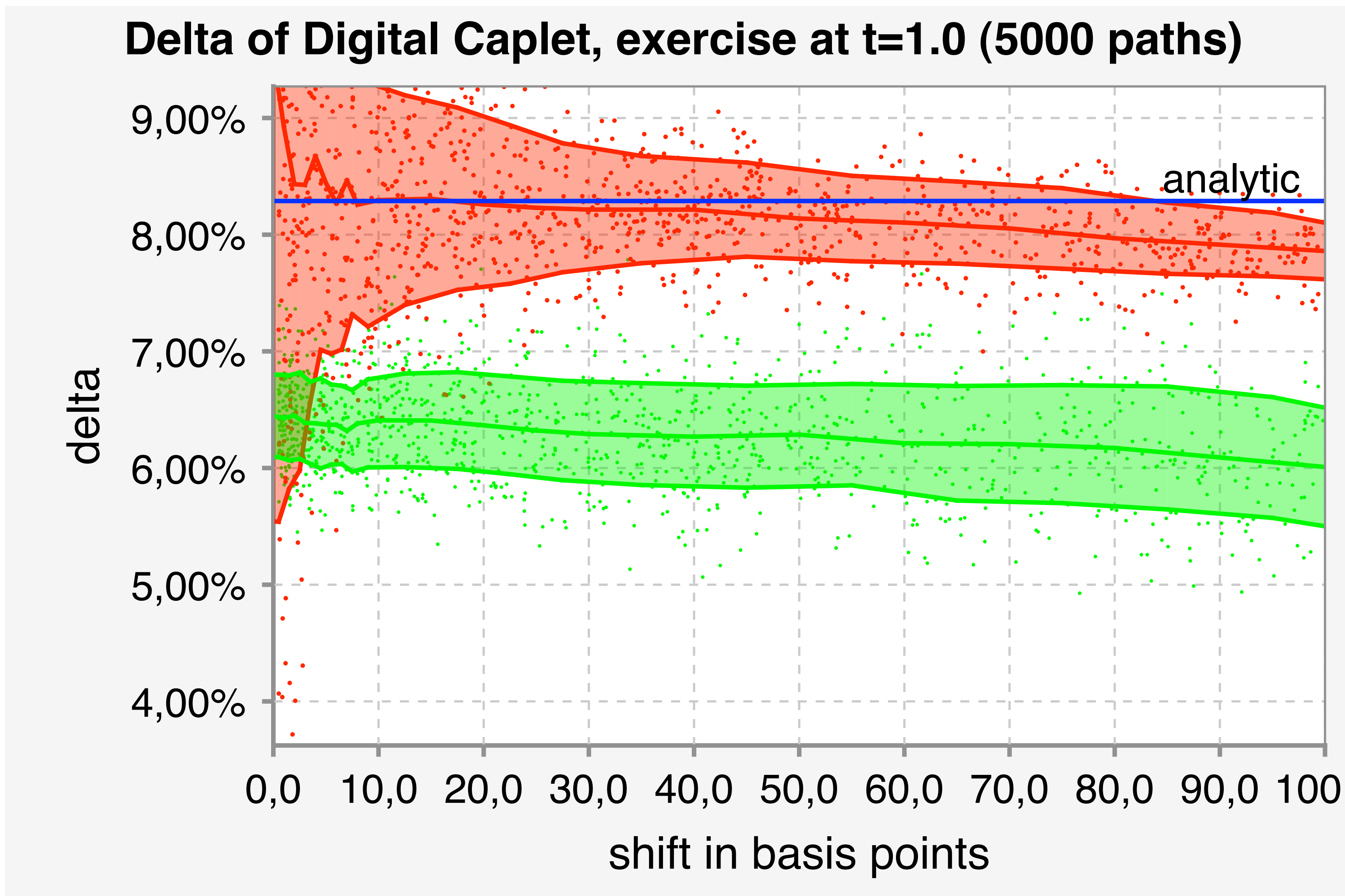
Drawback: Possible Loss if Mass for Short Maturities and Degenerate Diffusions



Drawback: Possible Loss if Mass for Short Maturities and Degenerate Diffusions



Drawback: Possible Loss if Mass for Short Maturities and Degenerate Diffusions



Drawback: Possible Loss if Mass for Short Maturities and Degenerate Diffusions

Solution 1: Decompose the shift into one part in the range of the diffusion and an orthogonal part. Use direct simulation for the latter part.

Soluation 2: Partial Proxy Simulation Scheme (next)

Partial Proxy Simulation Schemes

Partial Proxy Simulation Schemes

Partial Proxy Scheme: Consider *three* numerical schemes

$t \mapsto X(t)$	$t \in \mathrm{I\!R}$	model sde	
$t_i \mapsto X^0(t_i)$	$i = 0, 1, 2, \ldots$	scheme of unperturbed process X	$\rightarrow$ *reference scheme*
$t_i \mapsto X^*(t_i)$	$i = 0, 1, 2, \ldots$	scheme of perturbed process X_θ	$\rightarrow$ *target scheme*
$t_i \mapsto X^1(t_i)$	$i = 0, 1, 2, \ldots$	proxy scheme for X^*, see below	$\rightarrow$ *proxy scheme*

Idea: Simulate paths according to X^1, correct measure towards X^*.

Definition of X^1: Let f denote a given map $f : \mathrm{I\!R}^n \rightarrow \mathrm{I\!R}^k$ - the *proxy constraint*. Define the partial proxy scheme by

$$X^1(t_0) := X^*(t_0), \qquad X^1(t_{i+1}) := X^*(t_{i+1}) - \Gamma(t_i) \cdot v(t_i),$$

where $v(t_i)$ solves the proxy constraint

$$f(X^0(t_{i+1})) = f(X^*(t_{i+1}) - \Gamma(t_i) \cdot v(t_i)).$$

Thus we have

$$f(X^1(t_{i+1})) = f(X^0(t_{i+1})),$$

i.e. the quantity f will stay rigid.

Partial Proxy Scheme - Pricing:

Let $Y = (X(T_1), \ldots, X(T_m))$, $\quad Y^* = (X^*(T_1), \ldots, X^*(T_m))$, $\quad Y^1 = (X^1(T_1), \ldots, X^1(T_m))$.
We have $\mathrm{E}^{\mathbb{Q}}(f(Y(\theta)) \mid \mathcal{F}_{T_0}) \approx \mathrm{E}^{\mathbb{Q}}(f(Y^*(\theta)) \mid \mathcal{F}_{T_0})$ and furthermore

$$\begin{aligned}
\mathrm{E}^{\mathbb{Q}}(f(Y^*(\theta)) \mid \mathcal{F}_{T_0}) &= \int_\Omega f(Y^*(\omega, \theta)) \,\mathrm{d}\mathbb{Q}(\omega) = \int_{\mathbb{R}^m} f(y) \cdot \phi_{Y^*(\theta)}(y) \,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} f(y) \cdot \frac{\phi_{Y^*(\theta)}(y)}{\phi_{Y^1}(y)} \cdot \phi_{Y^1}(y) \,\mathrm{d}y = \mathrm{E}^{\mathbb{Q}}(f(Y^1) \cdot w(\theta) \mid \mathcal{F}_{T_0})
\end{aligned}$$

where $w(\theta) = \frac{\phi_{Y^*(\theta)}(y)}{\phi_{Y^1}(y)}$.

Note: Same idea as for (full) proxy scheme: We use the paths of X^1 and correct the measure.

Partial Proxy Scheme - Monte-Carlo Weights:

What is the change of measure from X^* to X^1?

From

$$X^1(t_{i+1}) \;:=\; X^*(t_{i+1}) - \Gamma(t_i) \cdot v(t_i),$$

we see that X^1 is just X^* where

$\Delta W(t_i)$ is replaced by $\Delta W(t_i) - v(t_i)$.

Thus

$$\underbrace{w(t_k) = \prod_0^{k-1} w_i}_{\text{For every time step...}} \quad \text{where} \quad w_i \;=\; \underbrace{\prod_j}_{\text{...and every indep. factor...}} \underbrace{\left.\frac{\exp\left(-\frac{(x_j - y_j - v_j(t_i))^2}{\Delta t_i}\right)}{\exp\left(-\frac{(x_j-y_j)^2}{\Delta t_i}\right)}\right|_{\substack{x_j = W_j(t_i), \\ y_j = W_j(t_{i+1})}}}_{\text{....calc. change in transition probability}}$$

$\Rightarrow$ Calculation of transition probability is simple. To do: Find appropriate mean shift $v(t_i)$.

Example: Digital Caplet

Consider a LIBOR market model, $L(t) = \exp(X(t))$, where X simulated by an Euler scheme with diffusion $\Gamma \Delta W$.

- Payout is discontinuous in $L_i(T_i)$.
- Define the proxy constraint as

$$L_i^1(t) \;=\; L_i^0(t) \qquad \text{for } T_{i-1} \leq t < T_i \tag{*}$$

  $\Rightarrow$ One LIBOR per time is forced to be rigid, especially all LIBORs seen upon their reset dates.

- Find a corresponding mean shift $v(t_i)$ of the Brownian increment $\Delta W(t_i)$ such that (*) holds with

$$X^1(t_{i+1}) \;:=\; X^*(t_{i+1}) - \Gamma(t_i) \cdot v(t_i),$$

  e.g.

$$v(t) = (v_1(t), 0, \ldots, 0) \quad \text{with} \quad v_1(t) \;=\; \frac{\log(L^0(T_i, T_{i+1}; t)) - \log(L^1(T_i, T_{i+1}; t))}{f_{1,i}} \quad \text{for } T_{i-1} < t \leq T_i.$$

In other words: For $t \in [T_{i-1}, T_i)$ shift the first factor such that the i-th LIBOR stays rigid.

Note: This constraint works for all products where the trigger is a function of the LIBORs upon their reset date, e.g. for a TARN on a LIBOR rate.

Partial Proxy Simulation Schemes

Example: TARN on CMS rate
Consider a LIBOR market model, $L(t) = \exp(X(t))$, where X simulated by an Euler scheme with diffusion $\Gamma \Delta W$.

- Payout is discontinuous in the CMS rate. The CMS rate is a non-linear function of the LIBORs.
- Define the proxy constraint as

$$CMS(X^1(t)) = CMS(X_i^0(t)) \qquad \text{for all } t \qquad (*)$$

$\Rightarrow$ CMS rate is forced to be rigid.

- Find a corresponding mean shift $v(t_i)$ of the Brownian increment $\Delta W(t_i)$ such that (*) holds with

$$X^1(t_{i+1}) := X^*(t_{i+1}) - \Gamma(t_i) \cdot v(t_i),$$

- We do not need to solve the non-linear proxy constraint (*) exactly. It is sufficient to calculate a linearization numerically (e.g. by finite differences).

The following simplification works: Shift the first factor such that a first order approximation of the CMS stays rigid upon its reset date.

$$v(t_i) = (v_1(t_i), 0, \ldots, 0) \quad \text{with} \quad v_1(t_i) = \frac{CMS(X^0(t_{i+1})) - CMS(X^*(t_{i+1}))}{\Delta CMS}$$

where $\Delta CMS := \frac{1}{\epsilon}(CMS(X^*(t_{i+1}) - \Gamma(t_i) \cdot \epsilon \vec{f_1}) - CMS(X^*(t_{i+1})))$.

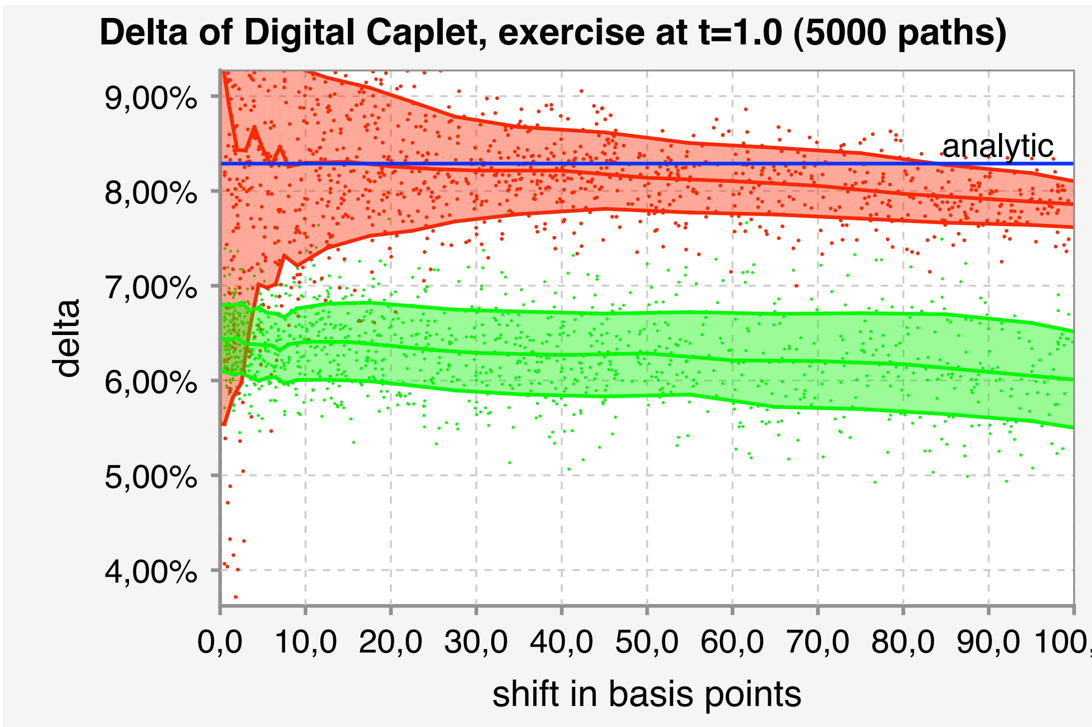
Numerical Results

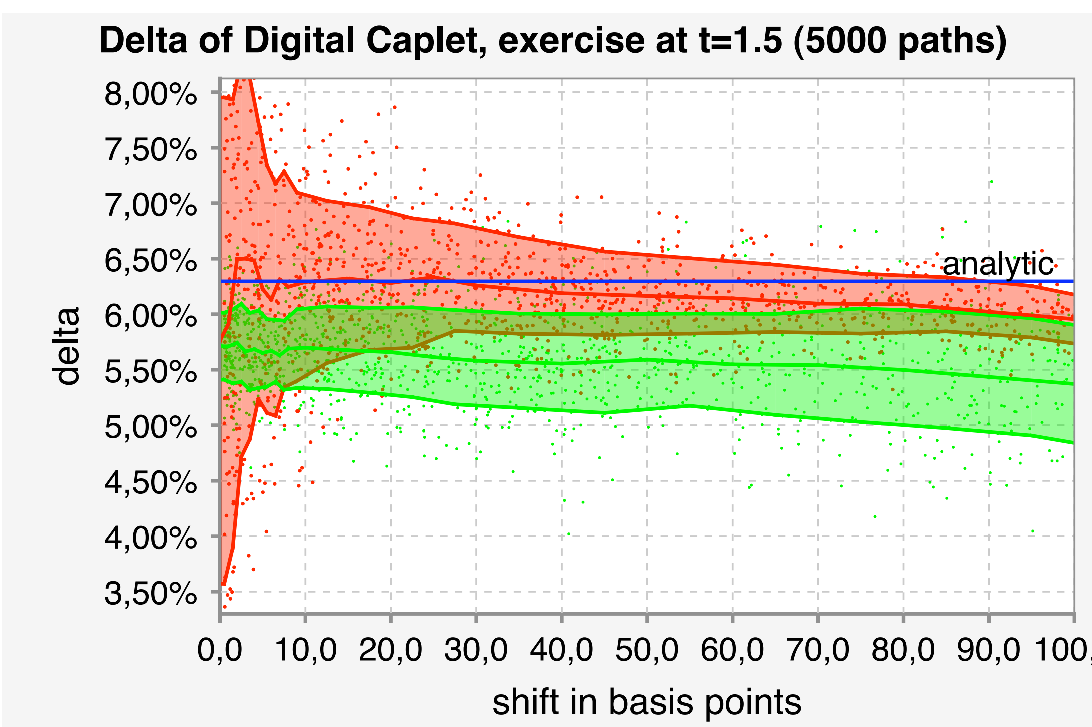
Digital Caplet

Comparisson with (full) Proxy Scheme

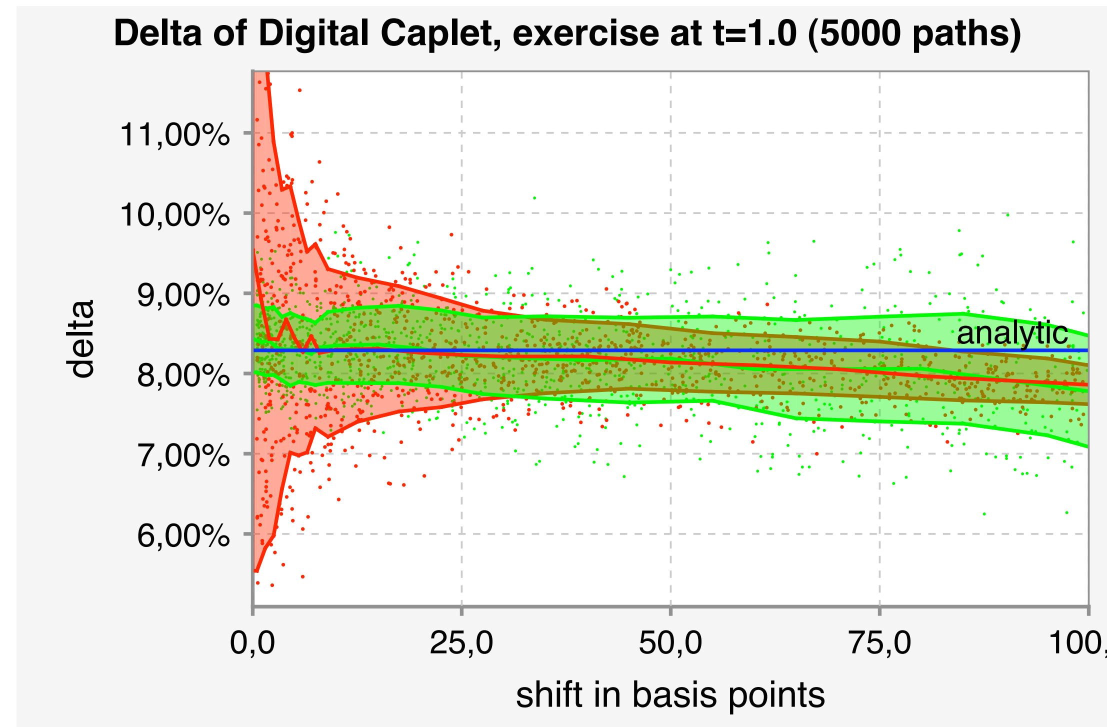
Partial Proxy Simulation Schemes: Digital Caplet

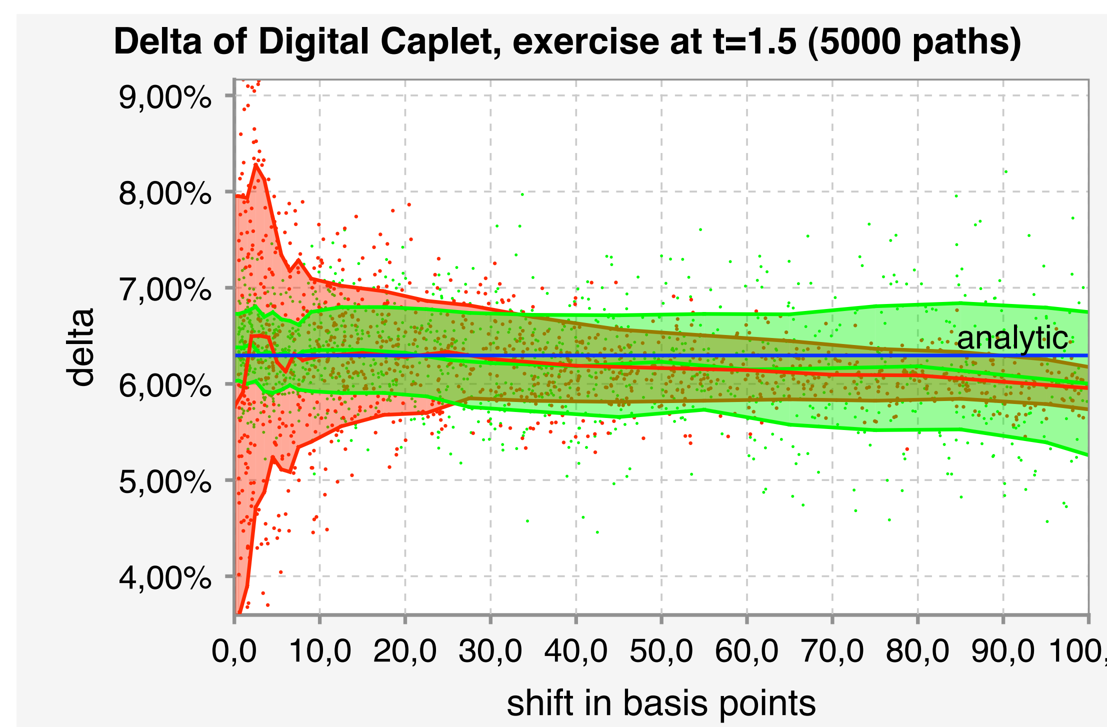
full proxy





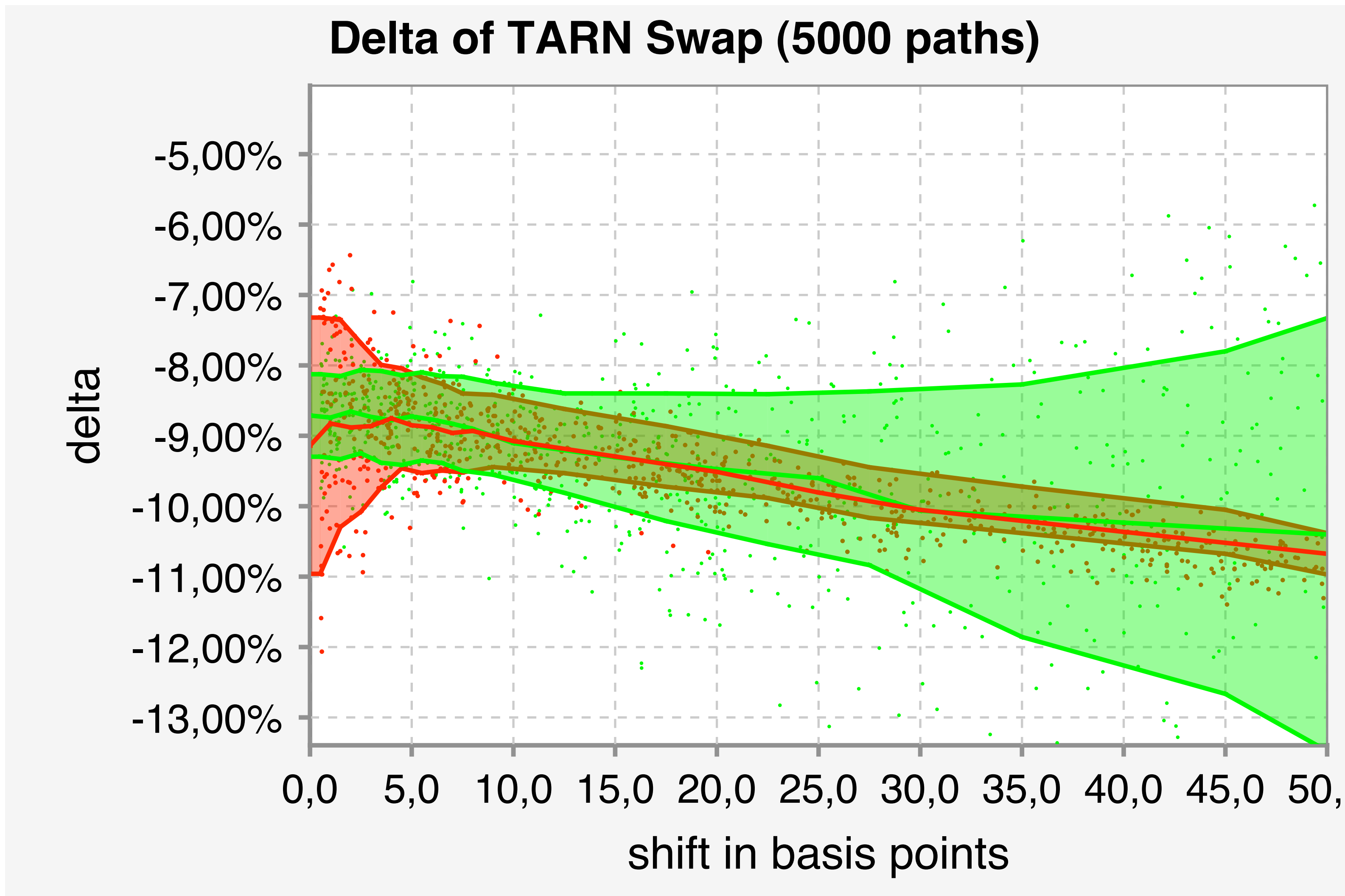
partial proxy (1-dim LIBOR constaint)



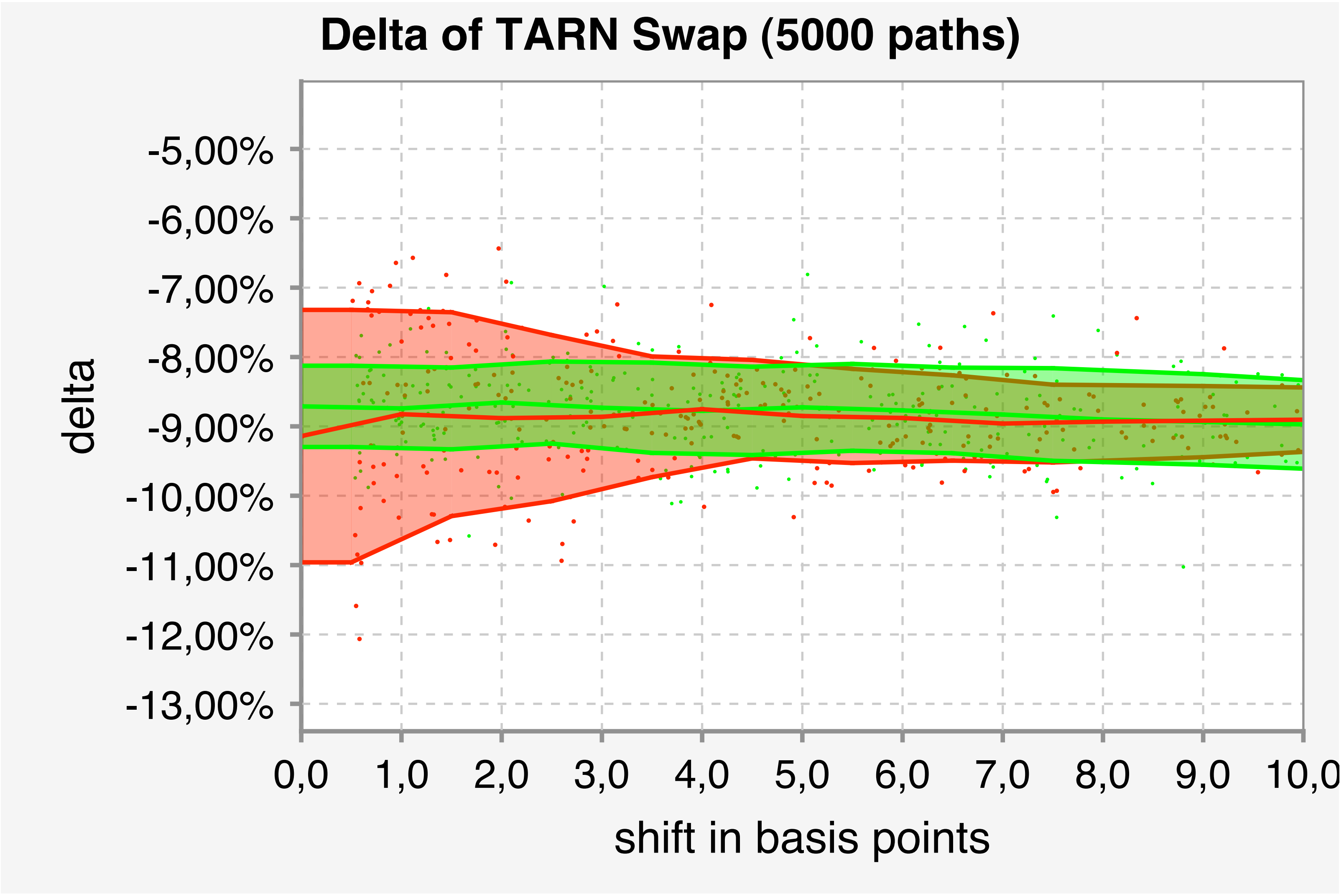


TARN

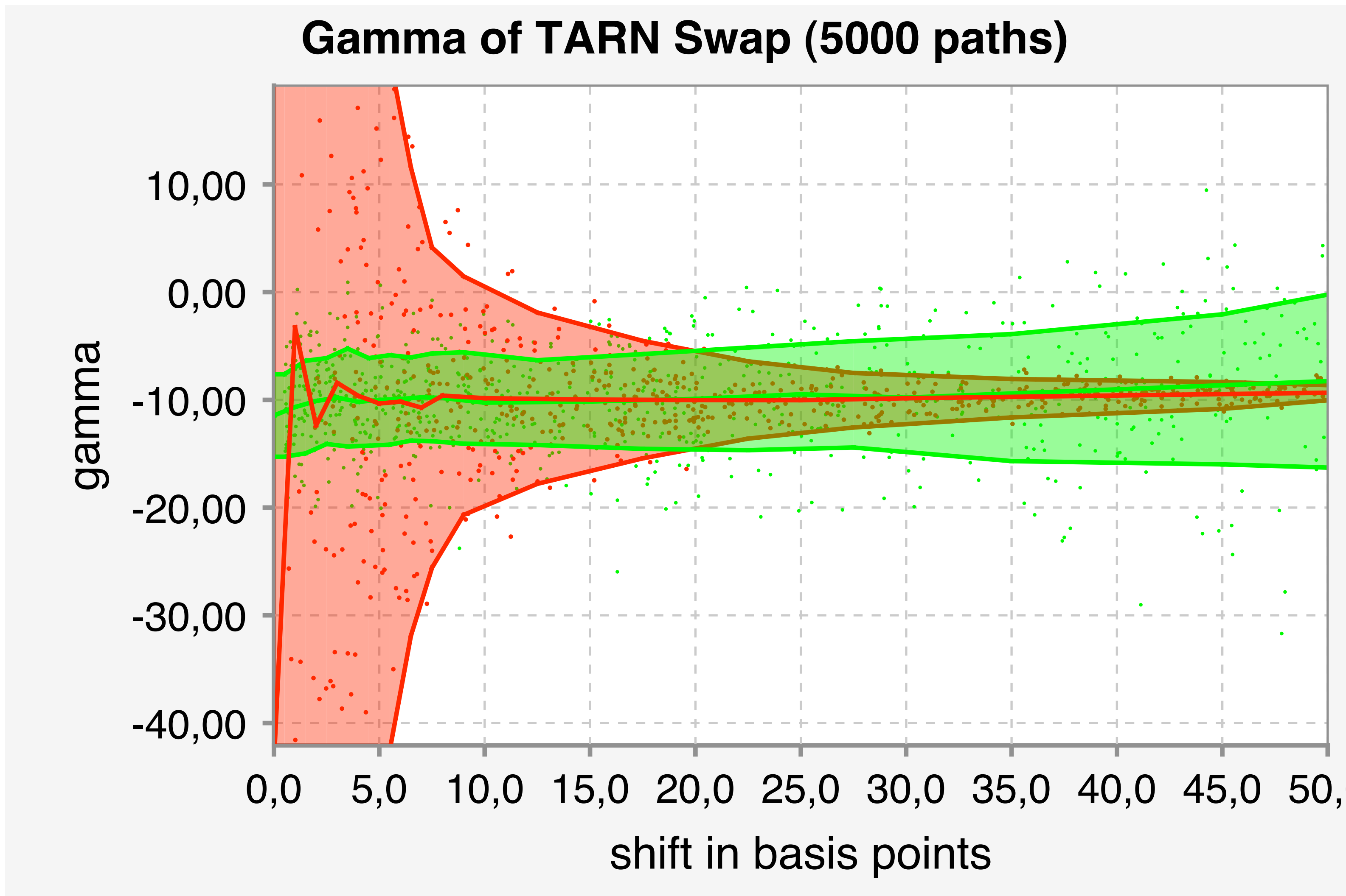
with LIBOR based index (reverse floater)



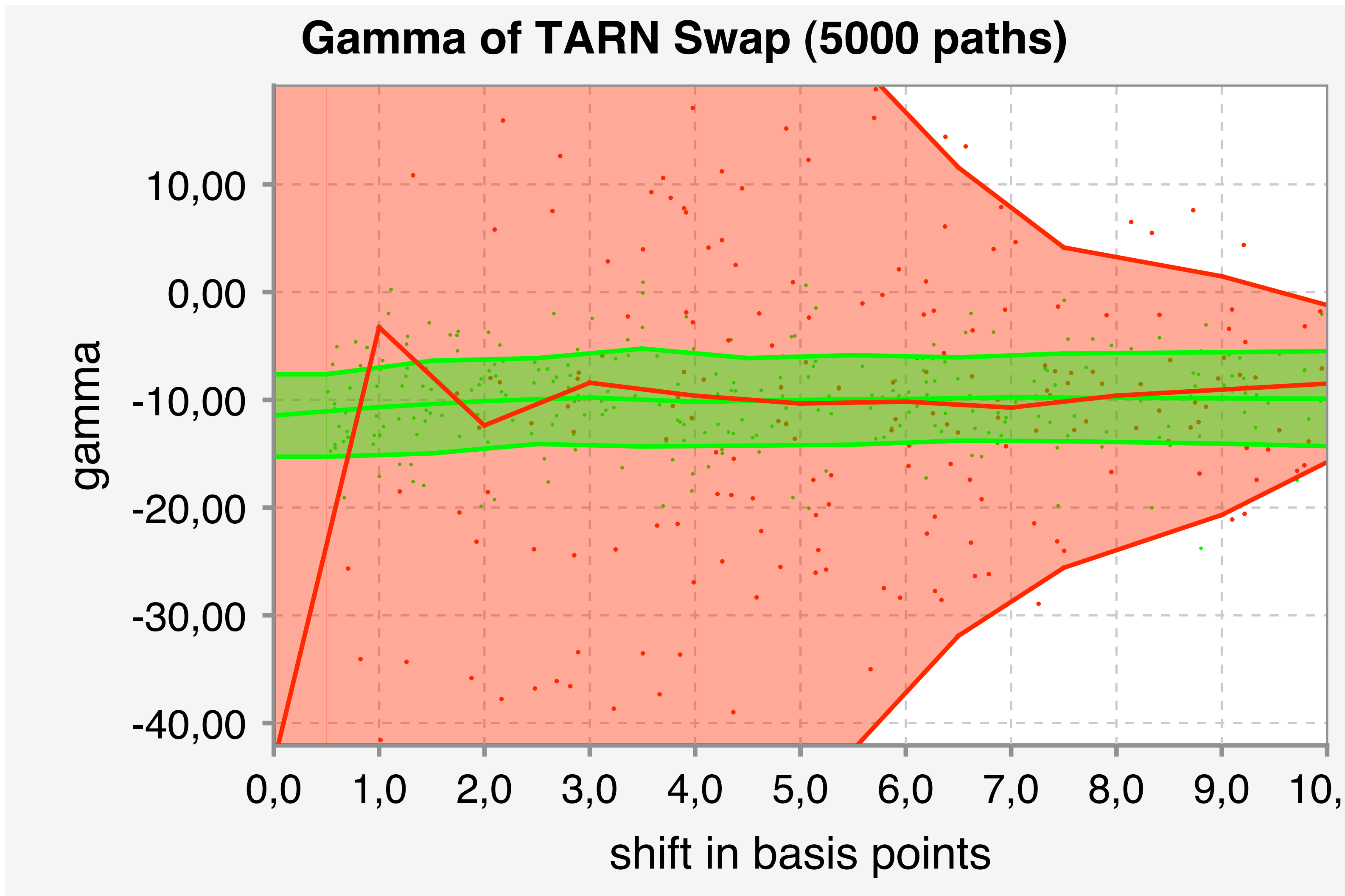
Proxy Constraint: LIBORs upon reset



Proxy Constraint: LIBORs upon reset



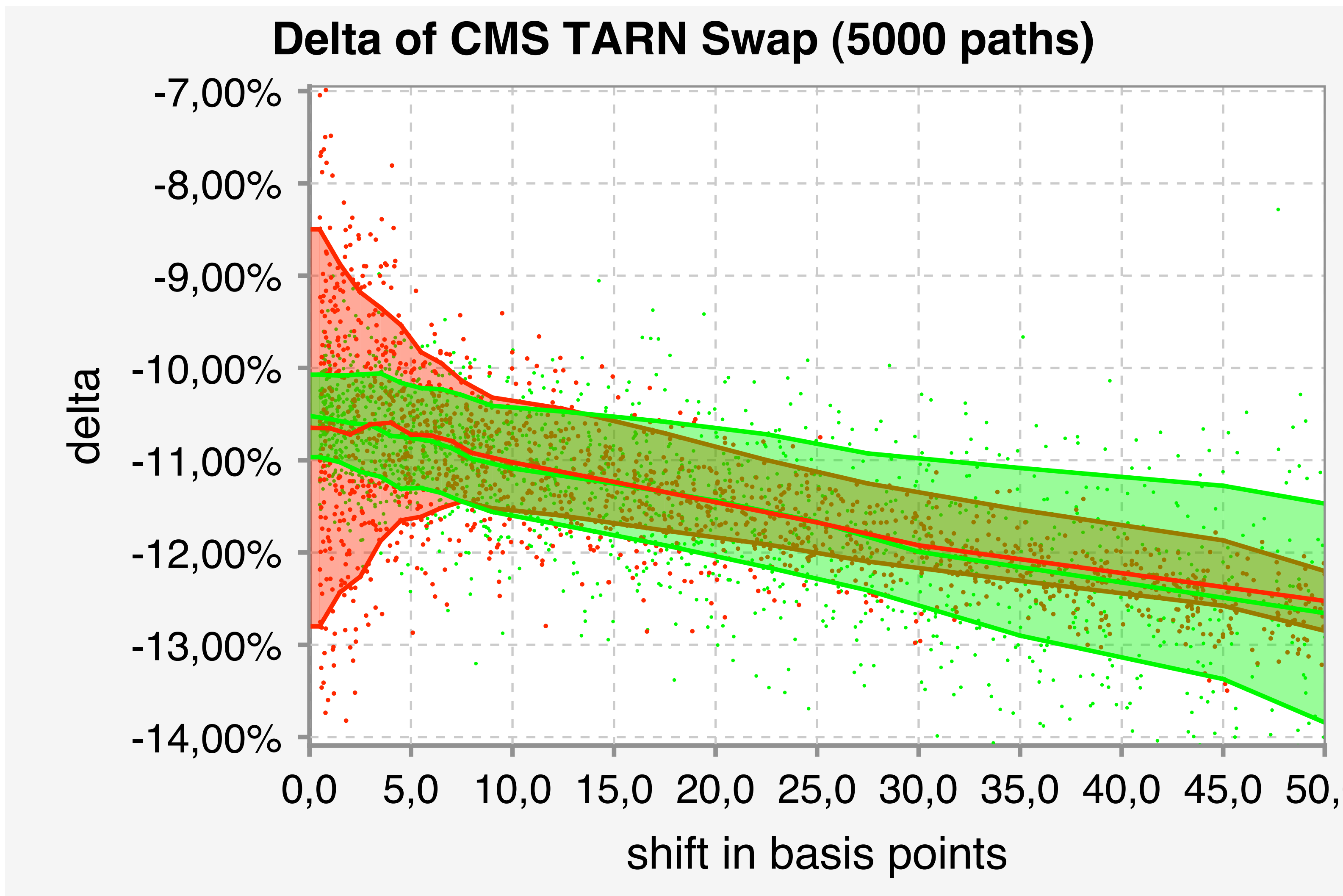
Proxy Constraint: LIBORs upon reset



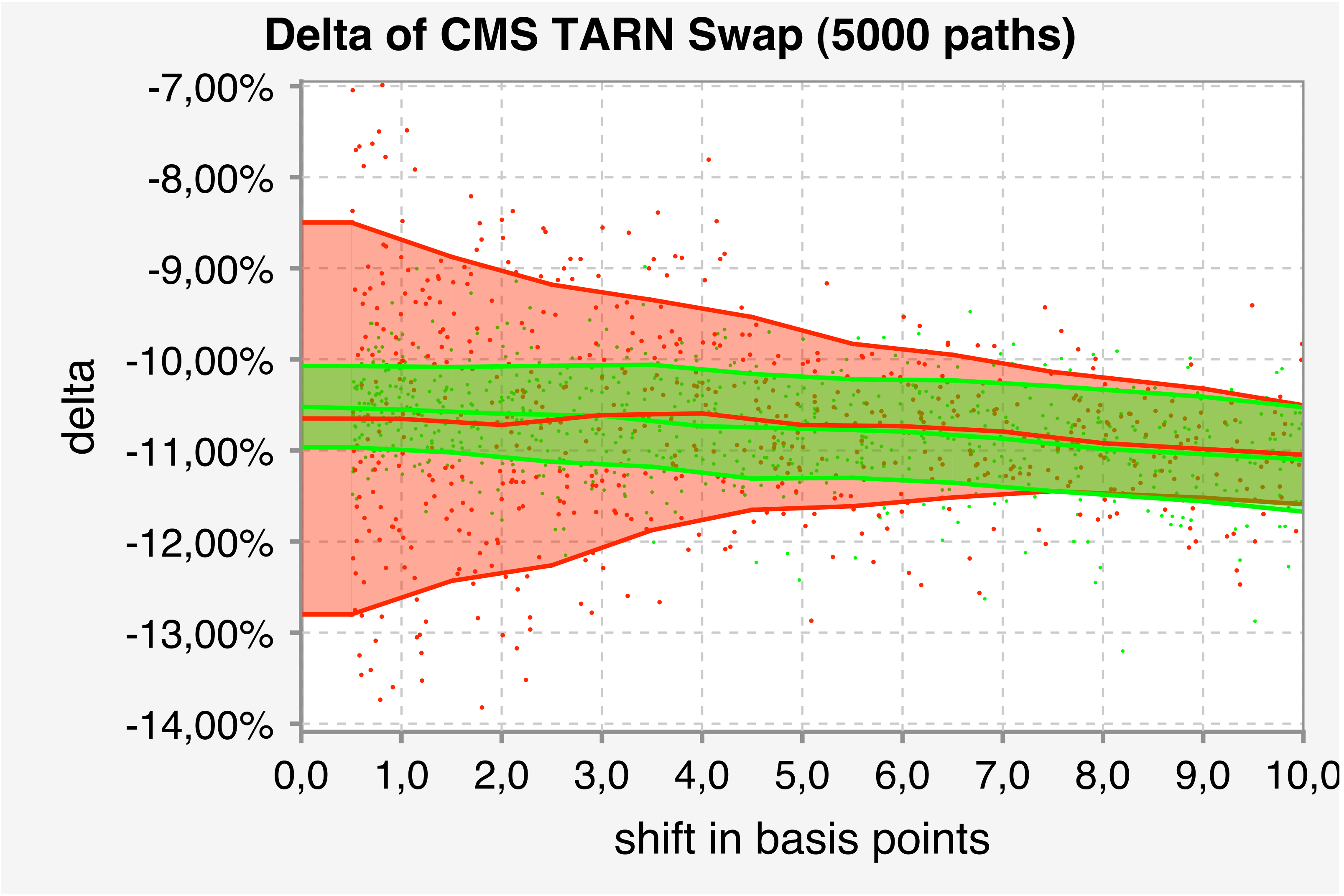
Proxy Constraint: LIBORs upon reset

TARN

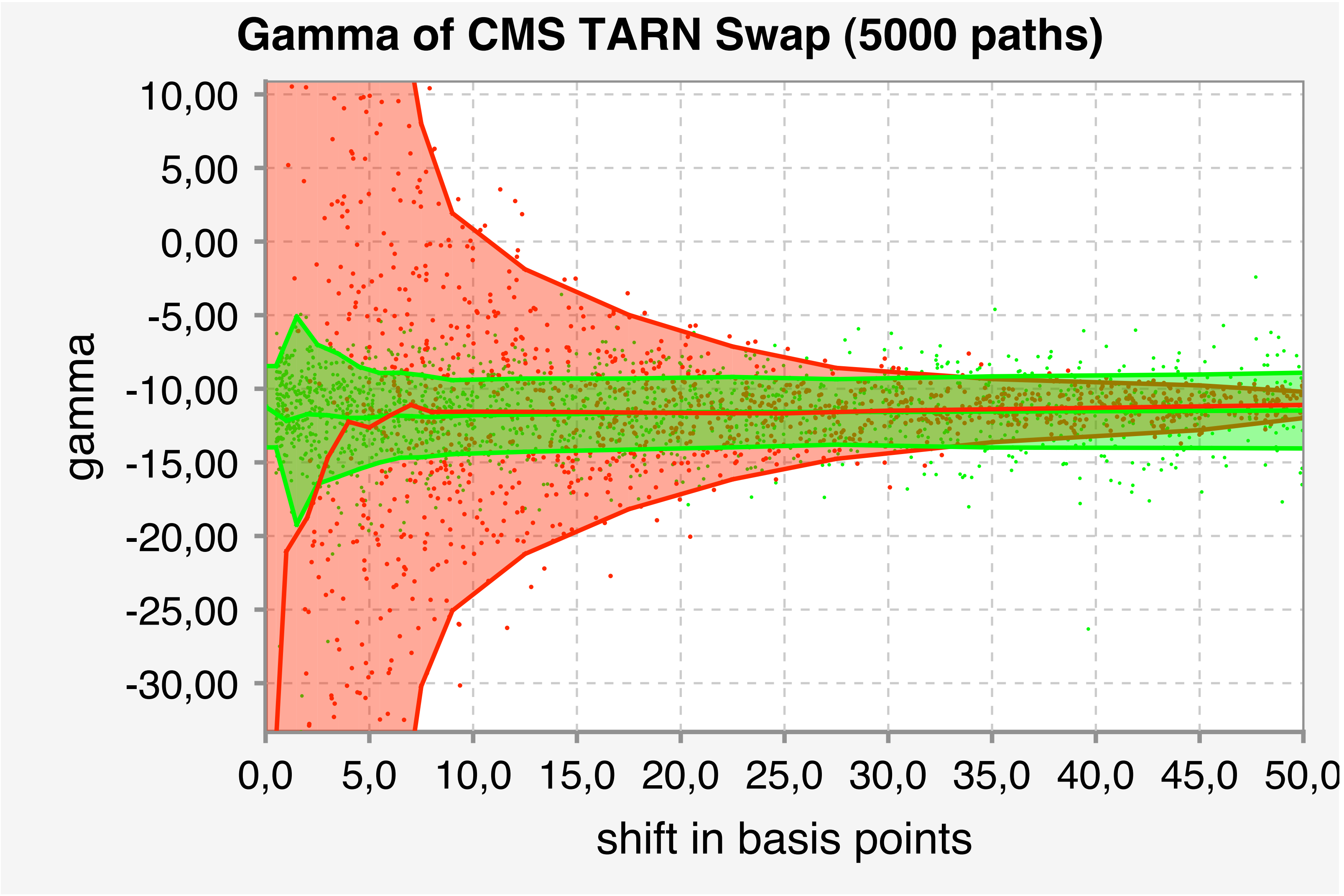
with CMS based index (reverse CMS floater)



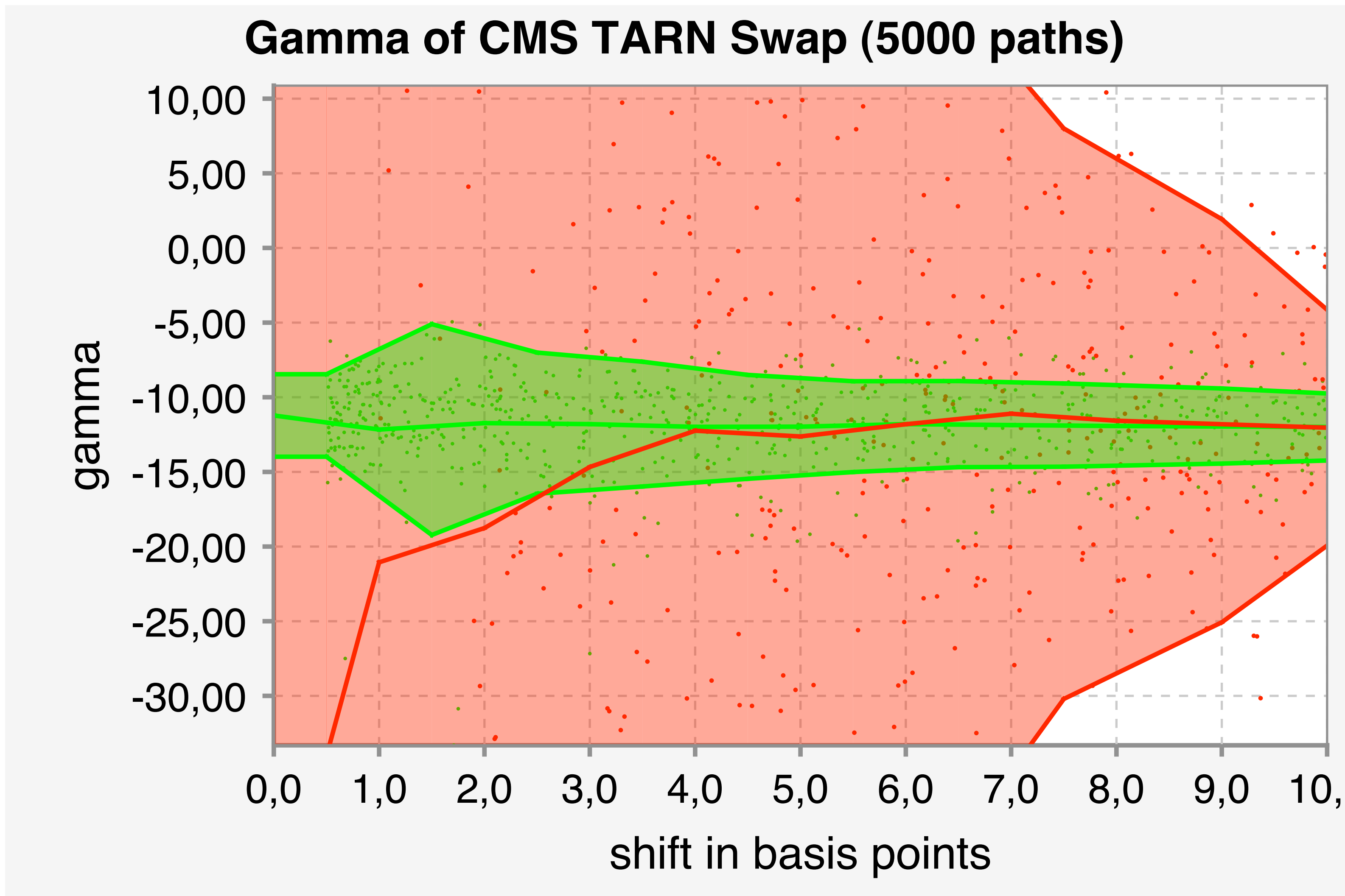
Proxy Constraint: Linearized CMS rate



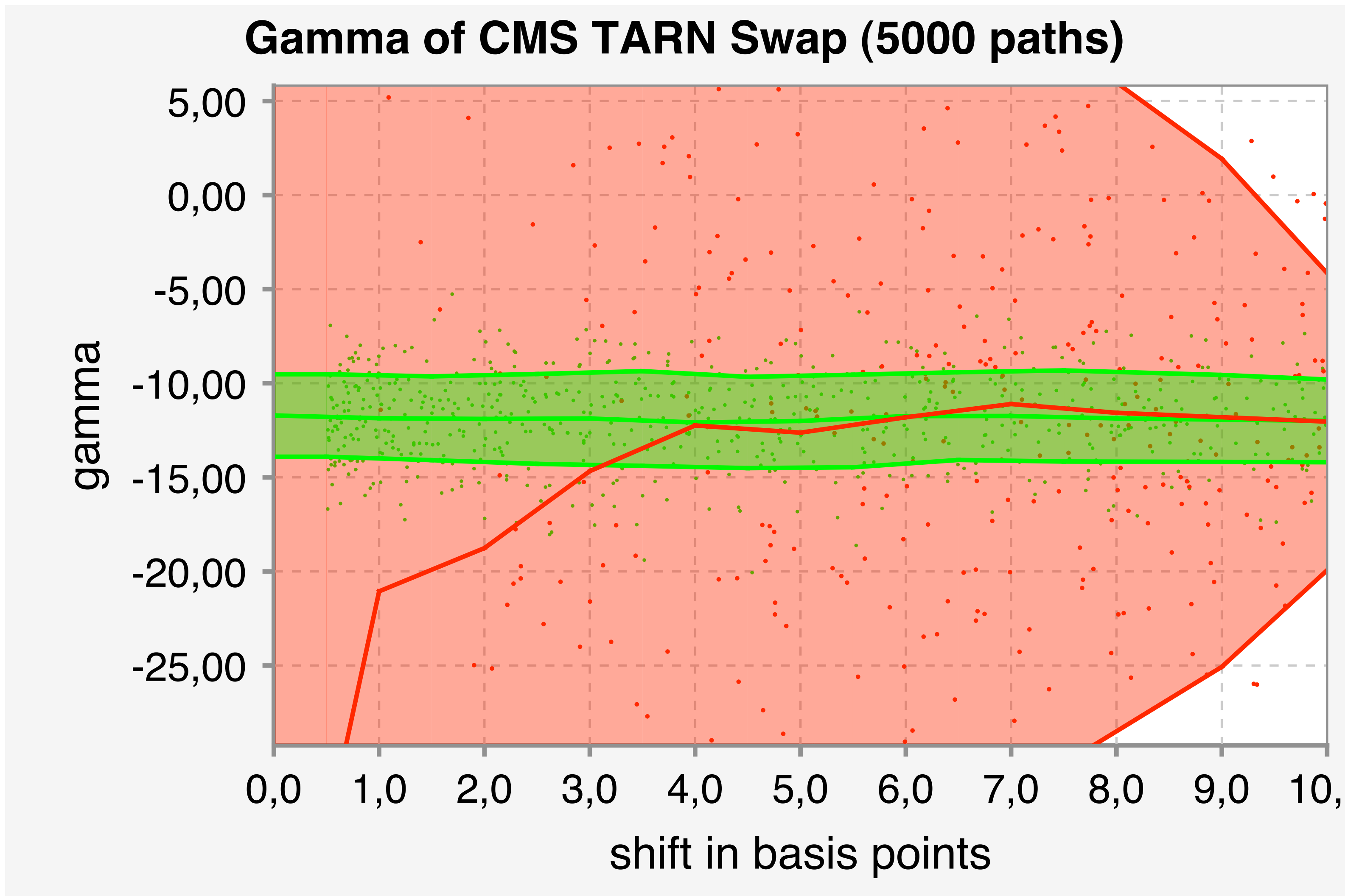
Proxy Constraint: Linearized CMS rate



Proxy Constraint: Linearized CMS rate



Proxy Constraint: Linearized CMS rate



Proxy Constraint: CMS rate (a few Netwon steps)

Appendix

Appendix: Quadratic WKB Expansion for the LMM Transition Probability Density

Three assumptions. First

(A) The operator L is uniformly parabolic in $\mathbb{R}^n$, i.e. there exists $0 < \lambda < \Lambda < \infty$ such that for all $\xi \in \mathbb{R}^n \setminus \{0\}$

$$0 < \lambda \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda. \tag{1}$$

(B) The coefficients of L are bounded functions in $\mathbb{R}^n$ which are uniformly Hölder continuous of exponent α ($\alpha \in (0,1)$).

guarantee that fundamental solution exists and is is strictly positive. The third assumption

(C) the growth of all derivatives of the smooth coefficients functions $x \to a_{ij}(x)$ and $x \to b_i(x)$ is at most of exponential order, i.e. there exists for each multiindex α a constant $\lambda_\alpha > 0$ such that for all $1 \leq i, j, k \leq n$

$$\left|\frac{\partial^\alpha a_{jk}}{\partial x^\alpha}\right|, \left|\frac{\partial^\alpha b_i}{\partial x^\alpha}\right| \leq \exp\left(\lambda_\alpha |x|^2\right), \tag{2}$$

guarentees (pointwise) convergence of coefficient functions $x \to c_k^y(x) := c_k(x,y)$ and $x \to d_k^y(x) := d_k(x,y)$ in the standard WKB-expansion

$$p(\delta t, x, y) = \frac{1}{\sqrt{2\pi\delta t}^n} \exp\left(-\frac{d^2(x,y)}{2\delta t} + \sum_{i\geq 0} c_i(x,y)\delta t^i\right). \tag{3}$$

and in the new WKB expansion (we call it the quadratic WKB expansion), which is

$$\begin{aligned} p(\delta t, x, y) &= \tfrac{1}{\sqrt{2\pi\delta t}^n} \times \\ \exp&\left(-\tfrac{\left(\sum_{i\geq 0} d_i \delta t^i\right)^2}{2\delta t} + \textstyle\sum_{i\geq 0}(c_i^y(y) + \nabla\left(c_i^y - \sum_{l=1}^{i-1} d_l^y d_{i-l}^y\right)(y)\cdot(x-y))\delta t^i\right). \end{aligned} \tag{4}$$

(This is from the ansatz

$$p(\delta t, x, y) = \frac{1}{\sqrt{2\pi\delta t}^n} \exp\left(-\frac{\left(\sum_{i\geq 0} d_i \delta t^i\right)^2}{2\delta t} + \sum_{i\geq 0}(\alpha_i^y + \beta_i^y \cdot (x-y))\delta t^i\right). \tag{5}$$

where $\cdot$ denotes the scalar product. Here α_i^y and β^y are affine terms depending on y (compensation terms).

From the target scheme only the transition probability is needed.

Kampen [KF] derived a quadratic WKB expansion for the LIBOR Market Model (see left).

This enables us to construct a proxy scheme simulation with almost arbitrary small time discretization error - even for a large time steps ΔT.

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References (1/2)

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please check http://www.christian-fries.de/finmath/talks/2006proxyScheme for updates